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The Sine-Cosine Wavelet and Its Application in the Optimal Control of Nonlinear Systems with Constraint

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ABSTRACT

In this paper, an optimal control of quadratic performance index with nonlinear constrained is presented. The sine-cosine wavelet operational matrix of integration and product matrix are introduced and applied to reduce nonlinear differential equations to the nonlinear algebraic equations. Then, the Newton-Raphson method is used for solving these sets of algebraic equations. To present ability of the proposed method, two classes, first order system and second order system, are considered. The obtained results show that the proposed method offers improved performance

1. Introduction

Nonlinear systems are very important because most of these systems have nonlinear dynamics. So, many research works and designers have showed an active interest in development and applications of nonlinear systems [1-3]. Orthogonal functions and polynomial series have received considerable attention in dealing with various problems of dynamical systems.

There are three sets of orthogonal functions which are widely used. The first one includes sets of piecewise constant basis functions (PCBFs) such as, Walsh, block pulse, etc. The second set consists of sets of orthogonal polynomials such as, Laguerre polynomials, Legendre polynomials, Chebyshev polynomials, etc. The third one is the widely used sets of sine–cosine functions in Fourier series [4]. Wavelets theory is a relatively new and emerging field in mathematical research. Wavelet theory [5] has been applied in a wide range of engineering science; particularly, wavelets are successfully used in signal analysis for waveform representation and segmentations, identification, optimal control and many other applications. Wavelets permit accurate representation of a variety of functions and operators. The examples are applying Legendre wavelets [6-9], sine–cosine wavelets [10, 11] and Chebyshev wavelets [12].

The main characteristic of this technique is that it reduces these problems to those of solving a system of algebraic equations and thus greatly simplifies the problem. In this investigation, the sine-cosine wavelets were used to find optimal control of nonlinear systems. By this numerical technique, a difficult problem was reduced to the straightforward nonlinear algebraic equations, which could be solved using a digital computer. Numerical examples were given to show accuracy of the technique. The paper is organized as follows. In order to provide a proper background, sine-cosine wavelets are explained in Section 2. Two optimal control problems are solved by sine-cosine wavelet using The Newton-Raphson51 method in

Section 3. Conclusion is presented in Section 4.

2. Description of Sine-Cosine Wavelets

2.1. Sine-Cosine Wavelets

Wavelets have been very successful in approximate solution of different types of systems. They constitute a family of functions constructed from dilation and translation of a single function, called the mother wavelet $\psi(x)$.

Sine-Cosine wavelets $\psi_{n,m}(t)=\psi(n,k,m,t)$ are defined as follows [11]:

$$\psi_{n,m}(t)=\begin{cases} 2^{\frac{k+1}{2}} f_m\left(2^k t-n\right) & \frac{n}{2^k}\le t\le \frac{n+1}{2^k} \\ 0 & \text{otherwise}\end{cases}$$

with

$$f_m(t)=\begin{cases} \frac{1}{\sqrt{2}} & m=0 \\ \cos(2m\pi t) & m=1,2,\ldots,L \\ \sin(2(m-L)\pi t) & m=L+1,L+2,\ldots,2L\end{cases}$$

$n=0,1,2,\ldots,2^{k-1}\quad k=0,1,2,\ldots$

where L is any positive integer.

2.2. Function Approximation

A function $f(t)\in L^2[0,1]$ can be approximated as:

$$f(t)=\sum_{m=0}^{2L}\sum_{n=0}^{2^k-1} C_{n,m}\psi_{n,m}(t) \tag{1}$$

where $C_{n,m}=\ <f(t),\psi_{n,m}(t)>$, in which $<\ .\ ,\ .\ >$ denotes the inner product as:

$$C_{n,m}=<f(t),\psi_{n,m}(t)>\ =\int_{-\infty}^{+\infty} f(t)\psi_{n,m}(t)dt \tag{2}$$

Eq. (1) can be written in a matrix form as:

$$f(t)=C^T\psi(t) \tag{3}$$

or

$$f(t)=\psi^T(t)C \tag{4}$$

where C and $\psi(t)$ are $2^k(2L+1)\times 1$ matrices which are given by:

$$C^T=\left[c_{0,0}\ \ c_{0,1}\ \cdots\ c_{0,2L}\ \ c_{1,0}\ \cdots\ c_{1,2L}\ \cdots\ c_{2^k-1,0}\ \cdots\ c_{2^k-1,2L}\right] \tag{5}$$

$$\psi(t)=\left[\psi_{0,0}(t)\quad \psi_{0,1}(t)\ldots\quad \psi_{0,2L}(t)\quad \psi_{1,0}(t)\ldots\psi_{2^k-1,0}(t)\ldots\psi_{2^k-1,2L}(t)\right] \tag{6}$$

2.3. Operational Matrix of Integration

Integration of the vector $\psi(t)$ defined in Eq. (6) can be written as:

$$\int_0^t \psi(s)ds=P_{\left(2^k(2L+1)\right)\times\left(2^k(2L+1)\right)}\psi_{\left(2^k(2L+1)\right)\times 1}(t) \tag{7}$$

Using Eqs.(3) and (7), $\int_0^t f(s)ds$ can be expressed as :

$$\int_0^t f(s)ds=\int_0^t C^T\psi(s)ds=C^T P\psi(t) \tag{8}$$

Also, using Eqs. (4) and (7), $\int_0^t f(s)ds$ can be defined as :

$$\int_0^t f(s)ds=\int_0^t \psi^T(s)Cds=\psi^T(t)P^T C \tag{9}$$

In (9), the matrix P is obtained as follows [11]:

$$P=\frac{1}{2^{k+\frac{1}{2}}}\begin{bmatrix} F & S & \cdots & S \\ 0 & F & \cdots & S \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F\end{bmatrix}_{2^k\times 2^k}$$

where, S and F are given by

$$S=\begin{bmatrix} \sqrt{2} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0\end{bmatrix}_{(2L+1)\times(2L+1)}$$

$$F=\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \cdots & 0 & \frac{-1}{\pi} & \frac{-1}{2\pi} & \cdots & \frac{-1}{L\pi} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{2}\pi} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{2\sqrt{2}\pi} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \frac{1}{L\sqrt{2}\pi} \\ \frac{1}{\pi} & \frac{-1}{\sqrt{2}\pi} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{2\pi} & 0 & \frac{-1}{2\sqrt{2}\pi} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{L\pi} & 0 & 0 & \cdots & \frac{-1}{L\sqrt{2}\pi} & 0 & 0 & \cdots & 0\end{bmatrix}_{(2L+1)\times(2L+1)}$$

2.4. THE PRODUCT OPERATIONAL MATRIX

The product operational matrix $\tilde{C}$ can be defined as follows:

$$\psi_i(t)\psi_i^T(t)C_i = \tilde{C}_i\psi_i(t) \quad (10)$$

for $i = 0,1,2,\dots,2^k-1$. In fact [6]

$$\psi\psi^T C \approx \tilde{C}\psi \quad (11)$$

where

$$\tilde{C} = \mathrm{diag}\left(\tilde{c}_0,\tilde{c}_1,\dots,\tilde{c}_{2^k-1}\right)$$

and $\tilde{C}$ is a $\left(2^k(2L+1)\right)\times\left(2^k(2L+1)\right)$ product operational matrix. To illustrate the calculation procedures, we choose $M=1$ and $k=1$, then, we have

$$C = \begin{bmatrix} c_{00} & c_{01} & c_{02} & c_{10} & c_{11} & c_{12}\end{bmatrix}^T$$

$$\psi(t) = \begin{bmatrix}\psi_{00} & \psi_{01} & \psi_{02} & \psi_{10} & \psi_{11} & \psi_{12}\end{bmatrix}^T \quad (12)$$

where

$$\left\{\begin{array}{c}\psi_{00} = \dfrac{2}{\sqrt{2}} \\ \psi_{01} = 2\cos\left(2\pi\left(2t-0\right)\right) \\ \psi_{02} = 2\sin\left(2\pi\left(2t-0\right)\right)\end{array}\right\} \qquad 0 \le t \le \frac{1}{2} \quad (13)$$

and

$$\left\{\begin{array}{c}\psi_{10} = \dfrac{2}{\sqrt{2}} \\ \psi_{11} = 2\cos\left(2\pi\left(2t-1\right)\right) \\ \psi_{12} = 2\sin\left(2\pi\left(2t-1\right)\right)\end{array}\right\} \qquad \frac{1}{2} \le t \le 1 \quad (14)$$

Using equation (12), we get

$$\psi(t)\psi^T(t) = \begin{bmatrix}
\psi_{00}\psi_{00} & \psi_{00}\psi_{01} & \psi_{00}\psi_{02} & \psi_{00}\psi_{10} & \psi_{00}\psi_{11} & \psi_{00}\psi_{12} \\
\psi_{01}\psi_{00} & \psi_{01}\psi_{01} & \psi_{01}\psi_{02} & \psi_{01}\psi_{10} & \psi_{01}\psi_{11} & \psi_{01}\psi_{12} \\
\psi_{02}\psi_{00} & \psi_{02}\psi_{01} & \psi_{02}\psi_{02} & \psi_{02}\psi_{10} & \psi_{02}\psi_{11} & \psi_{02}\psi_{12} \\
\psi_{10}\psi_{00} & \psi_{10}\psi_{01} & \psi_{10}\psi_{02} & \psi_{10}\psi_{10} & \psi_{10}\psi_{11} & \psi_{10}\psi_{12} \\
\psi_{11}\psi_{00} & \psi_{11}\psi_{01} & \psi_{11}\psi_{02} & \psi_{11}\psi_{10} & \psi_{11}\psi_{11} & \psi_{11}\psi_{12} \\
\psi_{12}\psi_{00} & \psi_{12}\psi_{01} & \psi_{12}\psi_{02} & \psi_{12}\psi_{10} & \psi_{12}\psi_{11} & \psi_{12}\psi_{12}
\end{bmatrix} \quad (15)$$

In Eq. (15), we have

$$\psi_{ij}\psi_{kl} = 0 \quad \text{if } i \ne k$$

Also, using equations (13) and (14), we get

$$\psi_{i0}\psi_{ij} = \frac{2}{\sqrt{2}}\psi_{ij}$$

$$\psi_{i1}\psi_{i1} = 2^2\cos^2\left(2\pi\left(2t-i\right)\right) = 4\left\{\frac{1+\cos\left(4\pi\left(2t-i\right)\right)}{2}\right\} = \frac{2}{\sqrt{2}}\psi_{i0} + \cdots$$

If we retain only the elements of equation (9), then we have

$$\psi\psi^T = \frac{2}{\sqrt{2}}\begin{bmatrix}
\psi_{00} & \psi_{01} & \psi_{02} & 0 & 0 & 0 \\
\psi_{01} & \psi_{00} & 0 & 0 & 0 & 0 \\
\psi_{02} & 0 & \psi_{00} & 0 & 0 & 0 \\
0 & 0 & 0 & \psi_{10} & \psi_{11} & \psi_{12} \\
0 & 0 & 0 & \psi_{11} & \psi_{10} & 0 \\
0 & 0 & 0 & \psi_{12} & 0 & \psi_{10}
\end{bmatrix}$$

in fact

$$\tilde{c}_i = \frac{2}{\sqrt{2}}\begin{bmatrix} c_{i0} & c_{i1} & c_{i2} \\ c_{i1} & c_{i0} & 0 \\ c_{i2} & 0 & c_{i0}\end{bmatrix}$$

3. OPTIMAL CONTROL PROBLEM

In this section, Sine-Cosine wavelets are applied for finding the following nonlinear optimal control examples.

3.1. THE FIRST-ORDER SYSTEMS

Example 1

Consider the optimal control problem of the fist-order nonlinear system [13]:

$$2\left[\dot{y}^2(t)\right] + \dot{y}(t) = u(t) \quad y(0) = 0.2, \quad 0 \le t \le 1 \quad (16)$$

With respect to a quadratic performance index:

$$J = \int_0^1 \left[y^2(t) + u^2(t)\right]dt \quad (17)$$

By integrating Eq. (16) from 0 to t and using Eqs. (3), (4), (7), (8), (9), and (11) the following can be given:

$$2\left[Y\tilde{Y}\psi(t) - \left(Y_0^2\right)^T\psi(t)\right] + \left[Y^T\psi(t) - Y_0^T\psi(t)\right] = U^T P\psi(t) \quad (18)$$

Eliminating $\psi(t)$ in equation (19) gives:

$$2Y\tilde{Y} + Y^T - U^T P - 2\left(Y_0^2\right)^T - Y_0^T = 0 \quad (19)$$

where Y_0^T and $\left(Y_0^2\right)^T$ are $1\times 2^k(2L+1)$ matrices given by:

$$Y_0^T = \begin{bmatrix}0.2 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots\end{bmatrix}$$

$$\left(Y_0^2\right)^T = \begin{bmatrix}0.04 & 0 & 0 & \cdots & \cdots & \cdots & \cdots\end{bmatrix}$$

For the performance Index, we have:

$$J = 10\int_0^1 \left[Y^T\psi(t)\psi^T(t)Y + U^T\psi(t)\psi^T(t)U\right]dt \quad (20)$$

$$J = 10\times Y^T LY + 10\times U^T LU \quad (21)$$

where

$$L = \int_0^1 \left[\psi(t)\psi^T(t) \right] dt \quad (22)$$

By minimizing Eq. (21) subject to Eq. (19) using the Lagrange multiplier technique, the following equation is obtained:

$$J^* = J + \lambda \left[2Y^T \tilde{Y} + Y^T - U^T P - 2\left(Y_0^2\right)^T - Y_0^T \right]^T \quad (23)$$

where λ is a $1 \times 2^k (2L+1)$ matrix as follows:

$$\lambda = \left[\lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4 \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \lambda_{2^k(2L+1)} \right]$$

The necessary conditions for minimization are:

$$\frac{\partial J^*}{\partial y_{nm}} = 0 \ , \frac{\partial J^*}{\partial u_{nm}} = 0 \qquad n = 1,2,\ldots,2^{k-1} \ , m = 0,1,\ldots,M-1 \quad (24)$$

and

$$\frac{\partial J^*}{\partial \lambda} = 0 \quad (25)$$

By this technique, Eq. (23) turns into a set of nonlinear algebraic equations which can be solved using the Newton-Raphson method [14] to obtain *J*. The approximated values of J are given in Table 1 and *y(t)* is shown in Fig. 1.

TABLE 1
APPROXIMATED VALUES OF *J* USING THE SINE-COSINE WAVELETS FOR FIRST ORDER SYSTEM

Sine-Cosine Wavelets	Approximated value of *J*
K=2, *L*=2	0.035499
K=3, *L*=3	0.035299
K=4, *L*=4	0.035220

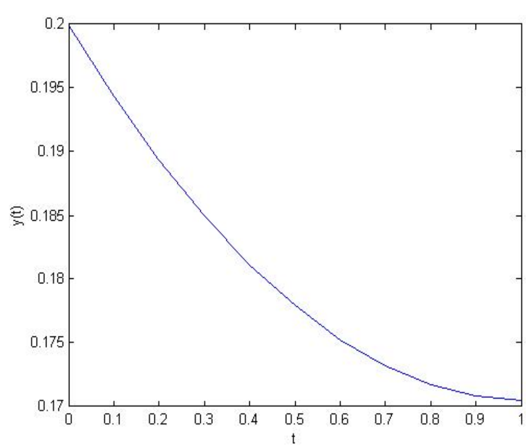

Figure 1: Response of the first order system

3.2. THE SECOND-ORDER SYSTEMS

Example 2

Consider optimal control problem of the second-order nonlinear system [13] as:

$$y(t)\ddot{y}(t) + \dot{y}(t) + y^2(t) = u(t) \ , \ y(0) = 10 \ , \dot{y}(0) = 0 \quad (26)$$

which performance index is considered the same as example 1 Let us use the following assumption:

$$\ddot{y}(t) = Y^T P(t) \quad (27)$$

Integrating Eq. (27) from 0 to t gives:

$$\dot{y}(t) - \dot{y}(0) = Y^T H P(t) \quad (28)$$

Integrating Eq. (28) from 0 to t gives:

$$y(t) - y(0) = Y^T H^2 P(t) \quad (29)$$

Expanding $y(0)$ using the Sine-Cosine wavelets is as:

$$y(0) = f^T P(t) \quad (30)$$

where f^T is a $1 \times m$ matrix given by:

$$f^T = [10 \quad 0 \quad 0 \quad \cdots \quad 0]$$

Substituting Eq. (30) into Eq. (29) gives

$$y(t) = Y^T H^2 P(t) + f^T P(t) \quad (31)$$

Expanding $u(t)$ using the Sine-Cosine wavelets can be written as:

$$u(t) = U^T P(t) \quad (32)$$

Substituting Eqs. (27), (31), (32) into Eq. (26) gives:

$$\begin{aligned} & Y^T H^2 P(t) P(t)^T Y + f^T P(t) P(t)^T Y + \\ & Y^T H P(t) + Y^T H^2 P(t) P(t)^T \left(H^2\right)^T Y + f^T P(t) P(t)^T f \\ & + Y^T H^2 P(t) P(t)^T f + f^T P(t) P(t)^T \left(H^2\right)^T Y = U^T P(t) \end{aligned} \quad (33)$$

Let us define:

$$Y^T H^2 = Z^T \quad (34)$$

Substituting Eq. (33) and Eq. (11) into Eq. (32) and eliminating $P(t)$ gives:

$$Z^T \tilde{Y} + S^T \tilde{Y} + f^T \tilde{Y} + Y^T H + Z^T \tilde{Z} + f^T \tilde{f} + Z^T \tilde{f} + f^T \tilde{Z} = U^T \quad (35)$$

Therefore, Eq. (35) turns into a set of nonlinear algebraic equations. Using the Newton-Raphson method which is applied in example 1, Y^T , λ ,U^T are obtained. In this case, Y^T are the coefficients of $\ddot{y}(t)$. In order to find the coefficients of $y(t)$, By solving the Eq. (31), we get the following equation:

$$\text{cofficeint of } y(t) = Y^T H^2 + f^T \quad (36)$$

Now, the minimum value of J is obtained. The approximated values of *J* is given in Table 2 and $y(t)$ is shown in Fig. 2.

TABLE 2
APPROXIMATED VALUES OF J USING THE SINE-COSINE WAVELETS FOR SECOND ORDER SYSTEM

Sine-Cosine Wavelets	Approximated value of J
K=2, L=2	0.035499
K=3, L=3	0.035299
K=4, L=4	0.035220

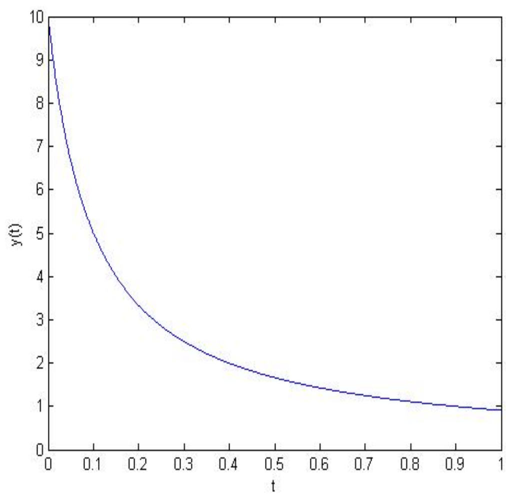

Figure 2: Response of the second order system

4. CONCLUSION

In this paper, a new method was presented based on wavelet theorem and using the sine-cosine wavelets and the Newton-Raphson method. Using the proposed method, nonlinear differential equations were converted into a set of nonlinear algebraic equations, which could be simply solved by a digital computer. Since operational matrix of integration and the product matrix contain many zero entries, it shows lower computational complexity with respect to the other approximations.Two numerical examples, such as first order system and second order system, were considered to show accuracy and applicability of the technique. As the final remark, the method can be extended for optimal control of nonlinear time varying systems.

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BIOGRAPHIES

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M. M. Farsangi was born in Kerman city, Iran country in 1972. She received her B.S. degree in Electrical Engineering from Ferdowsi University, Iran in 1995 and PhD degree in Electrical Engineering from Brunel Institute of Power Systems, Brunel University, UK in 2003. She is interested to robust control, model predictive control, power system control and computational intelligence. Dr. Farsangi joined the Department of Electrical Engineering, Shahid Bahonar University of Kerman, as an Assistant Professor, in 2003, and she has been an Associate Professor, since 2011.