

ABCD Matrix for Reflection and Refraction of Laser Beam at Tilted Concave and Convex Elliptic Paraboloid Interfaces and Studying Laser Beam Reflection from a Tilted Concave Parabola of Revolution

Mojtaba Mansour Abadi¹, Zabih Ghassemlooy^{1,*}, David Smith², and Wai Pang Ng¹

¹Optical Communications Research Group (NCRLab), Northumbria University, NE1 8ST, UK

²Microwave Imaging Research Group, Northumbria University, NE1 8ST, UK

*Corresponding Author's Information: z.ghassemlooy@northumbria.ac.uk

ARTICLE INFO

ARTICLE HISTORY:

Received 04 March 2015

Revised 22 March 2015

Accepted 22 March 2015

KEYWORDS:

Laser Beam

Gaussian Beam

ABCD Matrix

Reflection

Refraction

Tilted Surface

Convex/Concave Elliptic
Paraboloid Surface

ABSTRACT

Studying Gaussian beam is a method to investigate laser beam propagation and ABCD matrix is a fast and simple method to simulate Gaussian beam propagation in different mediums. Of the ABCD matrices studied so far, reflection and refraction matrices at various surfaces have attracted a lot of researches. However, in the previous work the incident beam and the principle axis of surface are in parallel. As an extension to those investigations, a general scheme that the incident beam is oblique is discussed here and the full analysis of the reflection and refraction of a Gaussian beam at the surface of a tilted concave/convex elliptic paraboloid surface is addressed. Based on the optical phase matching, analytic mathematical equations are derived for the spot size and the wavefront radius of a beam. Expressions are converted into the ABCD matrices, which are more convenient and practical to use. Finally, a practical case is analyzed by applying the obtained formulas. This analysis is important since paraboloid surfaces in optics or terahertz waves are used as mirrors or lenses.

1. INTRODUCTION

A solution of paraxial Helmholtz differential equation is Gaussian beam [1]-[2]. Gaussian beam has been applied in different areas like laser propagation model [3]-[4], fiber coupling model [5]-[6], wave expansion [7]-[8], free space optics propagation [9]-[10], scattering analysis [11]-[12], and synthesis and analysis of reflectors [13]-[14] and dielectric lenses [3], [15]. When a laser beam is reflected from a mirror or passing through a lens, it is essential to understand the laser beam characteristics before and after the optics, then in this paper reflection and refraction of a laser beam at a curved surface is studied.

As mentioned before, in most areas it is essential to investigate the beam propagation in different

environments and by using the ABCD matrix it is easier to simulate the Gaussian beam propagation [1]. Therefore, several researches have deduced the ABCD matrix for different mediums, especially for reflection and refraction of a Gaussian beam at parabola of revolution [16]-[19], elliptic paraboloid [17], ellipsoidal surface [18], and hyperboloid of revolution [16]-[19]. However, to the best of our knowledge none of the published papers considered a tilted surface/beam. Referring to [20]-[21], once can realize that using off-axis parabolic surfaces in optical antenna or terahertz wave applications, is inevitable. However, in published papers [16]-[19], the authors assumed that the beam is propagating in parallel to the surface principle axis. In [17], the authors have considered a paraxial beam reflected and refracted

from a convex elliptic paraboloid. Though in their analysis there is an angle between the beam propagation axis and the surface principle axis, during the mathematical manipulations the angle was neglected due to paraxial approximation.

In this paper, a simple analytical method is introduced to deal with tilted surfaces. One of the applications where this analysis is useful is propagation of a laser beam or terahertz wave through a tilted parabolic lens or reflection of a laser beam from a tilted parabolic mirror [22]-[23]. To investigate different cases of surface tilting angle, we consider two different types of paraboloid interface (concave and convex). For each case, separate spot size, and wave front radius are obtained by applying the electromagnetic boundary conditions along interface, and then ABCD matrix is extracted. At the end of each case, the input incident angle is assumed to be equal to Brewster angle and the distinct ABCD matrix for that special problem is presented. Though the investigation is based on Gaussian beam theory of lasers, it should be mentioned that as long as the following assumptions are correct, the obtained equations can be used for different Gaussian beam applications.

Three assumptions are made to simplify the mathematics. First, compared to radiuses of curvature of the wave front and optical surface, a small beam spot size is considered. With this assumption, up to the second order spatial variation which is important in the wave front change along the transverse coordinate is considered [18]. Second, for simplicity, the principle axis of the interface must lie in the plane of incidence. The third hypothesis requires that the beam propagation axis on the incident plane must intercept the interface principle axis. This assumption is more relevant for practical problems.

The rest of the paper is organized as follow. In Section 2, the original problem is simplified into an equivalent one. Section 3 analyzes the reflection of Gaussian beam at the surface of a concave elliptic paraboloid surface. Section 4 deals with refraction and Section 5 represents the problem geometry and principles for an elliptic convex paraboloid. Section 6 is devoted to study two interesting problems. In this section, first, we find a suitable point for a laser source in front of tilted parabola of revolution reflector which will result in a collimated beam. Then, we study the relation between angle of tilting and reflected beam waist. In the last section, the conclusion is summarized. All the obtained equations are summarized in appendix I.

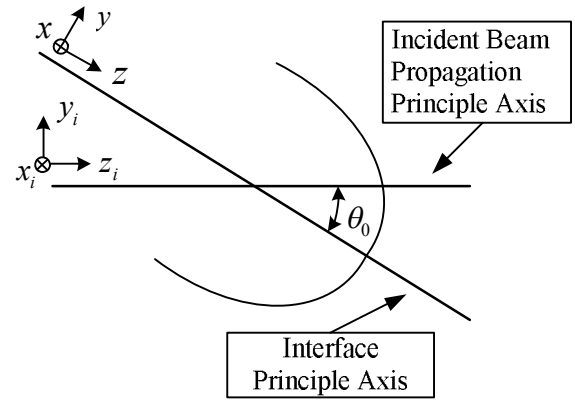
2. PROBLEM SIMPLIFICATION

Consider the geometry illustrated in Fig. 1(a). The Gaussian beam in the incident coordinate i is

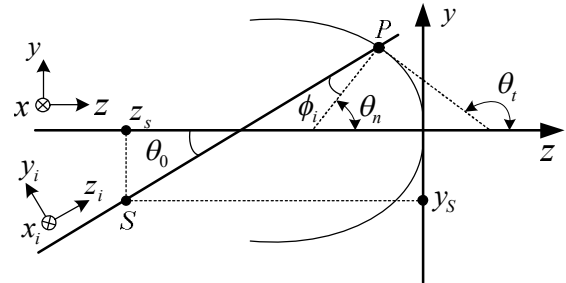
propagating along the z_i axis where the interface principle axis lies on z and θ_0 is the tilting angle.

The x and x_i axes are perpendicular to the plane of incidence and all coordinates are right-handed. In Fig. 1 the base coordinate is the incident coordinate and the interface coordinate is rotated clockwise by the angle of θ_0 . However, to avoid complexity, it is wise to consider the interface coordinate (x,y,z) as the base and rotate the incident coordinate counter-clockwise by the angle of θ_0 to obtain the equivalent geometry, Fig. 1(a).

As depicted in Fig. 1(b), the beam propagation axis passes the source point $S(x_s=0, y_s, z_s)$ in the interface coordinate. The elliptic concave paraboloid surface in the (x,y,z) coordinate is written as:



(a)



(b)

Figure 1: (a) Original problem geometry, (b) Equivalent problem geometry

$$\frac{x^2}{a} + \frac{y^2}{b} = -2z(a, b > 0) \quad (1)$$

where a and b define the curvature of the paraboloid in the x - z and y - z planes, respectively. Considering the beam propagation axis in the (x,y,z) coordinate as a line defined by:

$$y = \tan \theta_0 (z - z_s) + y_s \quad (2)$$

the interception point and the angle of tangential and normal lines in Fig. 1(b) are denoted by P , θ_t , and θ_n , respectively. The interception point on the incident

plane is described in the (x,y,z) coordinate as follow:

$$x_p = 0 \quad (3a)$$

$$y_p = -b \cot \theta_0 + \sqrt{b^2 \cot^2 \theta_0 + 2b(y_s \cot \theta_0 - z_s)} \quad (3b)$$

$$z_p = \cot \theta_0 (y_p - y_s) + z_s \quad (3c)$$

Note that with the assumption that the beam propagation axis lies on the incident plane, the expression $x_p = 0$ is obvious. Also, it is assumed that the interception point is in the $z < 0$ and $z > 0$ regions of coordinate. Therefore, based on the position of the source point coordinate and assuming that the incident point P is fixed, $\theta_0 > \theta_n$ or $\theta_0 < \theta_n$. Each case needs separate analysis then in Section 3.1 the case for $\theta_0 < \theta_n$ and in Section 3.2 the case for $\theta_0 > \theta_n$ are investigated.

3. REFLECTED BEAM

3.1: $\theta_0 < \theta_n$

Assuming $\theta_0 < \theta_n$, the equivalent problem is illustrated in Figure 2. The subscript i , r , and t are the incident, reflected and refracted (transmitted) beam coordinates respectively and in all coordinates the z axis is pointing to the propagation direction and both the y and z axes lie on the incident plane where the x axis is perpendicular to it. The coordinates are considered right-handed and the origins of all three coordinates lie at P . The incident angle is mentioned by ϕ_i .

From the literature the complex amplitude of a Gaussian beam can be written as [1]:

$$U_v(x_v, y_v, z_v) = \tilde{A}_v \exp(-j\Phi_v(x_v, y_v, z_v)) \quad (4)$$

where the subscript v refers to one of i , r , and t . $\Phi_v(\cdot)$ denotes the complex phase and has the following form [1].

$$\Phi_v(x_v, y_v, z_v) = k_v z_v + \frac{k_v}{2} \left(\frac{x_v^2}{q_{sv}} + \frac{y_v^2}{q_{tv}} \right) \quad (5)$$

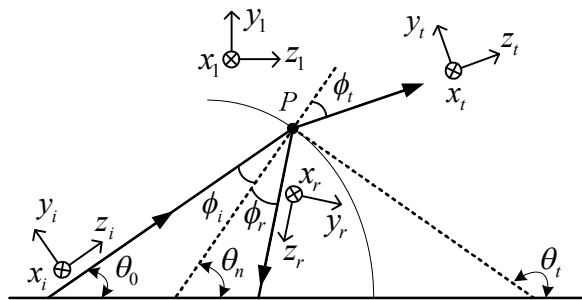


Figure 2: Coordinates for interface, and for incident, reflected and refracted beams. Reference unique coordinate for incident, reflected and refracted beams is presented too.

In (5), k_v is the wave number and defined by (6) and subscript S and T represent the sagittal and tangential planes, respectively. The parameter q will be defined later in the text.

$$k_v = \frac{2\pi n_v}{\lambda_0} \quad (6)$$

n_v and λ_0 in (6) are refractive index of the media and the free space wavelength. For the rest of the paper we assume the refractive index of n_1 and n_2 on the left and right sides of the interface, respectively.

By taking the derivative of (1) with respect to z on the z - y plane ($x=0$), the following expressions are obtained:

$$\tan \theta_t = \frac{-b}{y_p} \quad (7)$$

$$z_p = \frac{-0.5y_p^2}{b} \quad (8)$$

Note that for the geometry in Figure 1, $\theta_t = \theta_0 + \phi_i$. Then substituting (7) in (8) will result in the following equations.

$$\tan \theta_t = \frac{-b}{y_p} \quad (9a)$$

$$z_p = -0.5b \tan^2(\theta_0 + \phi_i) \quad (9b)$$

In the next step, the incident coordinate is transformed to a reference coordinate which will be used for applying boundary conditions for all beams. The reference coordinate is denoted by (x_1, y_1, z_1) and its origin is located at P and each axis in the reference coordinate is parallel to the corresponding axis in the interface coordinate (Figure 2). Using the following transformation, one can convert the incident coordinate (x_i, y_i, z_i) to the reference coordinate (x_1, y_1, z_1) .

$$x_i = x_1 \quad (10a)$$

$$y_i = +\cos \theta_0 y_1 - \sin \theta_0 z_1 \quad (10b)$$

$$z_i = +\sin \theta_0 y_1 + \cos \theta_0 z_1 \quad (10c)$$

On the other hand, the interface coordinate is converted to (x_1, y_1, z_1) coordinate by using:

$$x = x_1 \quad (11a)$$

$$y = y_1 + y_p = y_1 + b \tan(\theta_0 + \phi_i) \quad (11b)$$

$$z = z_1 + z_p = z_1 - 0.5b \tan^2(\theta_0 + \phi_i) \quad (11c)$$

Substituting (11) in (1) and solving for z_1 will yield:

$$z_1 = -\frac{x_1^2}{2a} - \frac{y_1^2}{2b} - y_1 \tan(\theta_0 + \phi_i) \quad (12)$$

Now, using the following sequence, the phase complex of the incident beam can be expressed in (x_1, y_1, z_1) coordinate:

$$\Phi_i(x_i, y_i, z_i) \xrightarrow{\text{using 10}} \Phi_i(x_1, y_1, z_1) \xrightarrow{\text{using 12}} \Phi_i(x_1, y_1) \quad (13)$$

Following (13), $\Phi_i(x_1, y_1)$ is expressed by:

$$\begin{aligned} \Phi_i(x_1, y_1) = & -k_i y_1 \frac{\sin \phi_i}{\cos(\theta_0 + \phi_i)} + k_i x_1^2 \left(\frac{1}{q_{Si}} - \frac{\cos \theta_0}{a} \right) \\ & + \frac{k_i y_1^2}{2} \left(\frac{1}{q_{Ti}} \times \frac{\cos^2 \phi_i}{\cos^2(\theta_0 + \phi_i)} \right. \\ & \left. - \frac{\cos \theta_0}{b} \right) + \sum f(x^m y^n), m, n \in \mathbb{Z}, \\ & m, n > 0, m + n > 2 \end{aligned} \quad (14)$$

Note that in (14), terms with variations of x_1 and y_1 greater than the second order are expressed in the summation term.

Referring to Figure 2, the phase complex of the reflected beam in reflection coordinate can be converted to the (x_1, y_1) coordinate by going through the sequence in (16).

$$x_r = x_1 \quad (15a)$$

$$y_r = -\cos(\phi_i + \phi_r + \theta_0) y_1 \quad (15b)$$

$$z_r = -\sin(\phi_i + \phi_r + \theta_0) y_1 \quad (15c)$$

$$- \cos(\phi_i + \phi_r + \theta_0) z_1 \quad (15c)$$

$$\Phi_r(x_r, y_r, z_r) \xrightarrow{\text{using 15}} \Phi_r(x_1, y_1, z_1) \quad (16)$$

$$\xrightarrow{\text{using 12}} \Phi_r(x_1, y_1) \quad (16)$$

Therefore $\Phi_r(x_1, y_1)$ would be:

$$\begin{aligned} \Phi_r(x_1, y_1) = & -k_r y_1 \frac{\sin \phi_r}{\cos(\theta_0 + \phi_i)} \\ & + k_r x_1^2 \left(\frac{1}{q_{Sr}} \right. \\ & \left. + \frac{\cos(\phi_i + \phi_r + \theta_0)}{a} \right) \\ & + \frac{k_r y_1^2}{2} \left(\frac{1}{q_{Tr}} \times \frac{\cos^2 \phi_r}{\cos^2(\theta_0 + \phi_i)} \right. \\ & \left. + \frac{\cos(\phi_i + \phi_r + \theta_0)}{b} \right) \\ & + \sum g(x^m y^n), m, n \in \mathbb{Z}, \\ & m, n > 0, m + n > 2 \end{aligned} \quad (17)$$

As in (14), the terms with variations of x_1 and y_1 greater than the second order are expressed in the summation term.

Based on the electromagnetic fields theory, the phases of incident, reflected and refracted (transmitted) beams must match exactly along the boundary interface, which means:

$$\begin{aligned} \Phi_i(\text{along the interface}) \\ = \Phi_r(\text{along the interface}) \\ = \Phi_t(\text{along the interface}) \end{aligned} \quad (18)$$

Neglecting high variations in (14) and (17) and using (18), the phase matching equations for the incident and reflected beams can be written. Equations (14) and (17) are expressed in the same coordinate and along the same interface, so (18) must be held for all (x_1, y_1) coordinates. It means that the coefficients of x_1 and y_1 variables with the same order

must be equal. Therefore, the following equations are obtained:

$$k_i \sin \phi_i = k_r \sin \phi_r \quad (19a)$$

$$k_i \left(\frac{1}{q_{Si}} - \frac{\cos \theta_0}{a} \right) = k_r \left(\frac{1}{q_{Sr}} + \frac{\cos(\phi_i + \phi_r + \theta_0)}{a} \right) \quad (19b)$$

$$\begin{aligned} \frac{k_i}{2} \left(\frac{1}{q_{Ti}} \times \frac{\cos^2 \phi_i}{\cos^2(\theta_0 + \phi_i)} - \frac{\cos \theta_0}{b} \right) \\ = \frac{k_r}{2} \left(\frac{1}{q_{Tr}} \times \frac{\cos^2 \phi_r}{\cos^2(\theta_0 + \phi_i)} + \frac{\cos(\phi_i + \phi_r + \theta_0)}{b} \right) \end{aligned} \quad (19c)$$

By replacing $k_i = k_r = 2\pi n_1 / \lambda_0$ in (19) and doing some simplification, (19) becomes:

$$\phi_i = \phi_r \quad (20a)$$

$$\frac{1}{q_{Sr}} = \frac{1}{q_{Si}} - \frac{1}{a} (\cos \theta_0 + \cos(2\phi_i + \theta_0)) \quad (20b)$$

$$\begin{aligned} \frac{1}{q_{Tr}} = \frac{1}{q_{Ti}} - \frac{\cos^2(\theta_0 + \phi_i)}{\cos^2 \phi_r} \\ \times \frac{1}{b} (\cos \theta_0 + \cos(2\phi_i + \theta_0)) \end{aligned} \quad (20c)$$

The first equation in (20) was expected from optical physics and the second and third ones are the sagittal and tangential relations for the reflected beam, respectively.

The complex beam parameter q_v is defined as:

$$\frac{1}{q_v} = \frac{1}{R_v} - j \frac{2}{k_v w_v^2} \quad (21)$$

In which R_v and w_v are the wave front radius of curvature and beam radius, respectively.

Using (21), the equations in (20) are written in the following forms:

$$w_{Sr} = w_{Si} \quad (22a)$$

$$\frac{1}{R_{Sr}} = \frac{1}{R_{Si}} - \frac{1}{a} (\cos \theta_0 + \cos(2\phi_i + \theta_0)) \quad (22b)$$

$$w_{Tr} = w_{Ti} \quad (22c)$$

$$\begin{aligned} \frac{1}{R_{Tr}} = \frac{1}{R_{Ti}} - \frac{\cos^2(\theta_0 + \phi_i)}{\cos^2 \phi_r} \\ \times \frac{1}{b} (\cos \theta_0 + \cos(2\phi_i + \theta_0)) \end{aligned} \quad (22d)$$

By defining a ray vector and ABCD matrix as in [1], the ABCD matrix form of (27) is presented by:

$$\begin{aligned} \begin{bmatrix} w_{Sr} \\ w_{Sr}/R_{Sr} \end{bmatrix} \\ = \begin{bmatrix} 1 & 0 \\ -\frac{1}{a} (\cos \theta_0 + \cos(2\phi_i + \theta_0)) & 1 \end{bmatrix} \begin{bmatrix} w_{Si} \\ w_{Si}/R_{Si} \end{bmatrix} \end{aligned} \quad (23a)$$

$$\begin{bmatrix} w_{Tr} \\ w_{Tr}/R_{Tr} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{\cos^2(\theta_0 + \phi_i)}{\cos^2 \phi_r} \times \frac{1}{b} (\cos \theta_0 + \cos(2\phi_i + \theta_0)) & 1 \end{bmatrix} \begin{bmatrix} w_{Ti} \\ w_{Ti}/R_{Ti} \end{bmatrix} \quad (23b)$$

3.2: $\theta_0 > \theta_n$

If $\theta_0 > \theta_n$, following the same procedure in Section 3.1 will results in:

a) The interface coordinate conversion:

$$x = x_1 \quad (24a)$$

$$y = y_1 + y_p = y_1 + b \tan(\theta_0 - \phi_i) \quad (24b)$$

$$z = z_1 + z_p = z_1 - 0.5b \tan^2(\theta_0 - \phi_i) \quad (24c)$$

b) Interface equation in the $z - y$ plane:

$$z_1 = -\frac{x_1^2}{2a} - \frac{y_1^2}{2b} - y_1 \tan(\theta_0 - \phi_i) \quad (25)$$

c) The transformation equation for converting the reflection coordinate in to (x_1, y_1, z_1) coordinate:

$$x_r = x_1 \quad (26a)$$

$$y_r = -\cos(\phi_i + \phi_r - \theta_0) y_1 + \sin(\phi_i + \phi_r - \theta_0) z_1 \quad (26b)$$

$$z_r = -\sin(\phi_i + \phi_r - \theta_0) y_1 - \cos(\phi_i + \phi_r - \theta_0) z_1 \quad (26c)$$

d) The complex phase of incident and reflected beams in (x_1, y_1) coordinates:

$$\begin{aligned} \Phi_i(x_1, y_1) = & +k_i y_1 \frac{\sin \phi_i}{\cos(\theta_0 - \phi_i)} \\ & + k_i x_1^2 \left(\frac{1}{q_{Si}} - \frac{\cos \theta_0}{a} \right) \\ & + \frac{k_i y_1^2}{2} \left(\frac{1}{q_{Ti}} \right. \\ & \times \frac{\cos^2 \phi_i}{\cos^2(\theta_0 - \phi_i)} - \frac{\cos \theta_0}{b} \Big) \\ & + \sum f(x^m y^n), m, n \in \mathbb{Z}, \\ & m, n > 0, m + n > 2 \end{aligned} \quad (27)$$

By equating the same order terms in (27) and neglecting the summation terms, the simplified form of boundary condition equations are obtained as follow:

$$k_i \sin \phi_i = k_r \sin \phi_r \quad (28a)$$

$$k_i \left(\frac{1}{q_{Si}} - \frac{\cos \theta_0}{a} \right) = k_r \left(\frac{1}{q_{Sr}} + \frac{\cos(\phi_i + \phi_r - \theta_0)}{a} \right) \quad (28b)$$

$$\begin{aligned} \frac{k_i}{2} \left(\frac{1}{q_{Ti}} \times \frac{\cos^2 \phi_i}{\cos^2(\theta_0 - \phi_i)} - \frac{\cos \theta_0}{b} \right) \\ = \frac{k_r}{2} \left(\frac{1}{q_{Tr}} \right. \\ \times \frac{\cos^2 \phi_r}{\cos^2(\theta_0 - \phi_i)} \\ \left. + \frac{\cos(\phi_i + \phi_r - \theta_0)}{b} \right) \end{aligned} \quad (28c)$$

The equations for ABCD matrices which are derived as in Section 3.1 are presented in Appendix I.

4. REFRACTED BEAM

4.1: $\theta_0 < \theta_n$

Referring to Figure 2, the phase complex of the refracted beam in the refracted coordinate can be converted to (x_1, y_1, z_1) coordinate by going through the sequence in (30).

$$x_t = x_1 \quad (29a)$$

$$y_t = +\cos(\phi_i - \phi_t + \theta_0) y_1 - \sin(\phi_i - \phi_t + \theta_0) z_1 \quad (29b)$$

$$z_t = +\sin(\phi_i - \phi_t + \theta_0) y_1 + \cos(\phi_i - \phi_t + \theta_0) z_1 \quad (29c)$$

$$\Phi_t(x_t, y_t, z_t) \xrightarrow[\text{using 12}]{\text{using 29}} \Phi_t(x_1, y_1, z_1) \xrightarrow{\text{using 12}} \Phi_t(x_1, y_1) \quad (30)$$

Then, the complex phase of the refracted beam would be as the following:

$$\begin{aligned} \Phi_t(x_1, y_1) = & -k_t y_1 \frac{\sin \phi_t}{\cos(\theta_0 + \phi_i)} \\ & + k_t x_1^2 \left(\frac{1}{q_{St}} \right. \\ & - \frac{\cos(\phi_i - \phi_t + \theta_0)}{a} \Big) \\ & + \frac{k_t y_1^2}{2} \left(\frac{1}{q_{Tt}} \right. \\ & \times \frac{\cos^2 \phi_t}{\cos^2(\theta_0 + \phi_i)} \\ & - \frac{\cos(\phi_i - \phi_t + \theta_0)}{b} \Big) \\ & + \sum h(x^m y^n), m, n \in \mathbb{Z}, \\ & m, n > 0, m + n > 2 \end{aligned} \quad (31)$$

Neglecting the summation terms in (31) and using (18) and (14), the boundary condition equations take the following form:

$$k_i \sin \phi_i = k_t \sin \phi_t \quad (32a)$$

$$k_i \left(\frac{1}{q_{St}} - \frac{\cos \theta_0}{a} \right) = k_t \left(\frac{1}{q_{St}} - \frac{\cos(\phi_i - \phi_t + \theta_0)}{a} \right) \quad (32b)$$

$$\begin{aligned} \frac{k_i}{2} \left(\frac{1}{q_{Ti}} \times \frac{\cos^2 \phi_i}{\cos^2(\theta_0 + \phi_i)} - \frac{\cos \theta_0}{b} \right) \\ = \frac{k_t}{2} \left(\frac{1}{q_{Tt}} \right. \\ \times \frac{\cos^2 \phi_t}{\cos^2(\theta_0 + \phi_i)} \\ \left. - \frac{\cos(\phi_i - \phi_t + \theta_0)}{b} \right) \end{aligned} \quad (32c)$$

Note that the first term in (32) is Snell's law. By replacing k_t with $\frac{2\pi n_2}{\lambda_0}$, the equation in (32) becomes:

$$\sin \phi_i = n \sin \phi_t \quad (33a)$$

$$\frac{1}{q_{St}} = \frac{1}{n} \times \frac{1}{q_{Si}} - \frac{1}{na} \left(\cos \theta_0 \right. \\ \left. - \cos(\phi_i + \theta_0) \sin \phi_i \right) \quad (33b)$$

$$\begin{aligned} \frac{1}{q_{Tt}} = \frac{n \cos^2 \phi_i}{n^2 - \sin^2 \phi_i} \times \frac{1}{q_{Ti}} - \frac{n \cos^2 \phi_i}{n^2 - \sin^2 \phi_i} \\ \times \frac{1}{b} \left(\cos \theta_0 \right. \\ \left. - \cos(\phi_i + \theta_0) \sin \phi_i \right) \end{aligned} \quad (33c)$$

where $n = n_2/n_1$.

Note that using the first equation in (28), ϕ_t is replaced by an expression of ϕ_i . The equations for ABCD matrices are presented in Appendix I.

By considering the condition of Brewster angle incidence ($\phi_B = \tan^{-1} n$) and letting ϕ_i be ϕ_B , it is possible to obtain expressions for refracted beam at $\phi_i = \phi_B$. The equations for ABCD matrices at $\phi_i = \phi_B$ are summarized in Appendix I.

4.2: $\theta_0 > \theta_n$

The procedure for this case is exactly the same as Section 4.1 except that the following transformation is used to convert the refracted coordinate to (x_1, y_1, z_1) .

$$x_t = x_1 \quad (34a)$$

$$y_t = +\cos(\phi_i - \phi_t - \theta_0) y_1 \\ + \sin(\phi_i - \phi_t - \theta_0) z_1 \quad (34b)$$

$$z_t = -\sin(\phi_i - \phi_t - \theta_0) y_1 \\ + \cos(\phi_i - \phi_t - \theta_0) z_1 \quad (34c)$$

Following the mentioned method, one can derive the equations summarized in Appendix I.

5. ELLIPTIC CONVEX PARABOLOID SURFACE

In the previous sections, it was assumed that the interface between mediums is in the shape of an elliptic concave paraboloid surface and explicit expressions for ABCD matrices were derived (Appendix I). In this section, the interface has the shape of an elliptic convex paraboloid surface.

The method of obtaining ABCD matrix is the same as Sections 3 and 4. However, in this case, there would be only one condition over θ_0 . Figure 3 depicts the geometry of the problem:

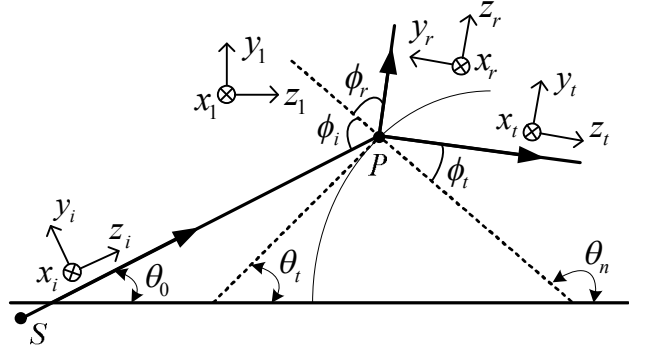


Figure 3: Coordinates for convex interface and for incident, reflected and refracted beams

The interface is expressed as:

$$\frac{x^2}{a} + \frac{y^2}{b} = +2z(a, b > 0) \quad (35)$$

And if the point P is fixed on the interface and the point S moves so that the angle of θ_0 is changed, there is no realistic solution for $\theta_0 > \theta_t$. Since for elliptic convex paraboloid interface $\theta_n > \theta_t$ then always $\theta_0 < \theta_n$. Then, it is not necessary to divide the problem into sub-problems.

After following the same procedure mentioned in Sections 3 and 4, one can derive the explicit equations of beam spot size and wave front radius of curvature for the sagittal and tangential planes. Also, by setting ϕ_i to ϕ_B , it is possible to formulate the ABCD matrix of the refracted beam at Brewster angle. The equations for this section are presented in Appendix I where the ABCD matrices for different geometries are summarized.

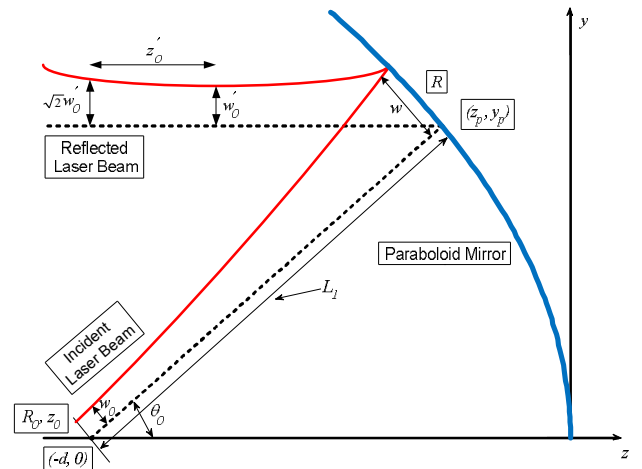


Figure 4: Laser beam and paraboloid mirror

6. STUDYING REFLECTION FROM A PARABOLOID MIRROR

In this section, reflection of a laser beam from a paraboloid mirror is investigated. Assume that the laser is placed in front of the mirror as illustrated in Figure 4.

In Figure 4, the laser is placed on the principle axis of the mirror and the distance between the mirror apex and laser beam waist is d . The tilting angle, Rayleigh range [1], and beam width of laser at source point is θ_0 , z_0 and w_0 , respectively, and w and R denote the beam width and the radius curvature of the incident beam at parabolic surface. Referring to [3] and Figure 4, one can calculate the distance between the incident point and laser source L_1 . Also, it is possible to calculate beam waist (w'_0) and Rayleigh range (z'_0) of reflected beam (Figure 4) as follow, respectively [1, 24]:

$$w'_{0(S,T)} = \frac{w}{\left[1 + (\pi w^2 / \lambda R_{(S,T)})^2\right]^{1/2}} \quad (36a)$$

$$z'_{0(S,T)} = \frac{\pi w^2}{\lambda \left[1 + (\pi w^2 / \lambda R_{(S,T)})^2\right]} \quad (36b)$$

where w and $R_{(S,T)}$ are beam width and radius of curvature at output plane of mirror respectively which are obtained by using the ABCD matrix for free space propagation and the ABCD matrix element from Appendix I and [1]:

$$w = w_0 \left[1 + \left(\frac{L_1}{z_0}\right)^2\right]^{1/2} \quad (37a)$$

$$\frac{1}{R_S} = \frac{1}{R} \left[-\frac{1}{a} [\cos \theta_0 + \cos(2\phi_i - \theta_0)]\right] \quad (37b)$$

$$\frac{1}{R_T} = \frac{1}{R} \left[-\frac{1}{b} \times \frac{\cos^2(\theta_0 - \phi_i)}{\cos^2 \phi_i} [\cos \theta_0 + \cos(2\phi_i - \theta_0)]\right] \quad (37c)$$

where R is obtained as [1]:

$$R = L_1 \left[1 + \left(\frac{z_0}{L_1}\right)^2\right] \quad (38)$$

Figure 5 illustrates the variations of w , w'_0 , and z'_0 with respect to d . Note that to plot these results the first assumption at the beginning of the paper is considered. To satisfy this condition, the results are plotted if $w \leq R/10$ and $w \leq R_{surf}/10$, where R_{surf} refers to radius curvature of the reflector. In both sagittal and tangential modes, it is clear that when the source point is located at the focal length of the mirror ($F = 0.05 \text{ m}$) the incident beam width and reflected

beam waists are identical, and Rayleigh range gets its maximum which means the reflected beam is quasi-collimated and also it is obvious that as the angle of tilting increases, beam width and Rayleigh range increase as well. So, it can be deduced that when the radius of curvature of the wave fronts is larger than beam spot size, and the principle axis of the parabola of revolution mirror lies in incident plane, the focal length of the mirror is fixed for different angles of incidence.

In the cases that in specific ranges of distance and angle, it is necessary to find the distance and angle at which the Rayleigh range is maximum, Rayleigh range in (36) can be written as:

$$z'_{0(S,T)}(\theta_0, d) = c \frac{w_0^2 + \left(\frac{w_0}{z_0}\right)^2 f^2(\theta_0, d)}{1 + c^2 \left[w_0^2 + \left(\frac{w_0}{z_0}\right)^2 f^2(\theta_0, d)\right]^2} \times \left[\frac{f(\theta_0, d)}{f^2(\theta_0, d) + z_0^2} + g_{S,T}(\theta_0, d)\right]^2 \quad (39)$$

where $c = \pi/\lambda$ is a constant and $f(\theta_0, d) = L_1$. The function $g_{S,T}(\theta_0, d)$ is defined as:

$$g_S(\theta_0, d) = -\frac{1}{a} [\cos \theta_0 + \cos(2\phi_i - \theta_0)] \quad (40a)$$

$$g_T(\theta_0, d) = -\frac{1}{b} \times \frac{\cos^2(\theta_0 - \phi_i)}{\cos^2 \phi_i} [\cos \theta_0 + \cos(2\phi_i - \theta_0)] \quad (40b)$$

It is possible to find the maximum of $z'_{0(S,T)}(\theta_0, d)$ using different numerical methods [25-27]. Using Genetic algorithm [25], the maximum Rayleigh range is obtained for the following parameters:

$$\begin{aligned} \theta_{0S} &= 90^\circ \\ d_S &= 0.05 \text{ m} \quad \text{for} \quad 1^\circ < \theta_{0(S,T)} < 90^\circ \\ \theta_{0T} &= 90^\circ \\ d_T &= 0.05 \text{ m} \quad \text{for} \quad 0.1 \times F < d_{(S,T)} < 2.0 \times F \end{aligned}$$

In another calculation, the incident angle θ_0 is set to 45° and the sagittal beam width of the reflected Gaussian beam regarding the distance up to 10 m between the mirror and the beam is calculated (Figure 6). Different values of d is considered to show the effect of source position on the beam width of reflected Gaussian beam. It is obvious that when the source is placed at the focal length of the mirror, the reflected beam is almost collimated whereas for other distances the beam diverges rapidly. Though this figure is just for sagittal plane, the same result is obtained for the tangential plane.

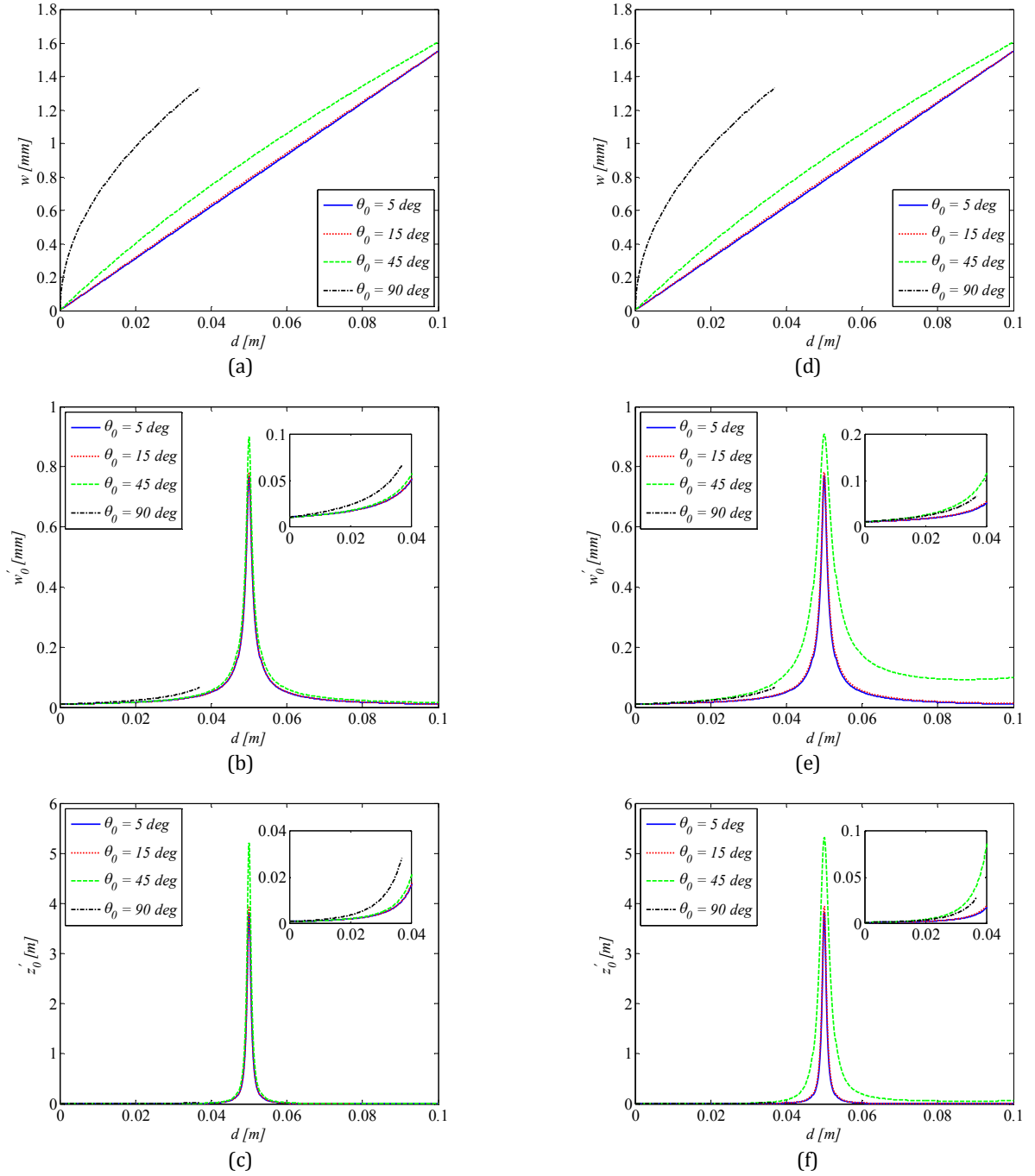


Figure 5: (a) Beam waist of incident beam at surface (sagittal), (b) Beam waist of reflected beam (sagittal), (c) Rayleigh range of reflected beam (sagittal), (d) Beam waist of incident beam at surface (tangential), (e) Beam waist of reflected beam (tangential), and (f) Rayleigh range of reflected beam (tangential) as a function of source distance to mirror apex. $a = b = 0.1$, $\lambda = 488\text{nm}$, and $w_0 = 10\mu\text{m}$. Different tilting angles are drawn with different line styles; $\theta_0 = 5^\circ$ solid line, $\theta_0 = 15^\circ$ dotted line, $\theta_0 = 45^\circ$ dashed line, and $\theta_0 = 90^\circ$ dash-dotted line.

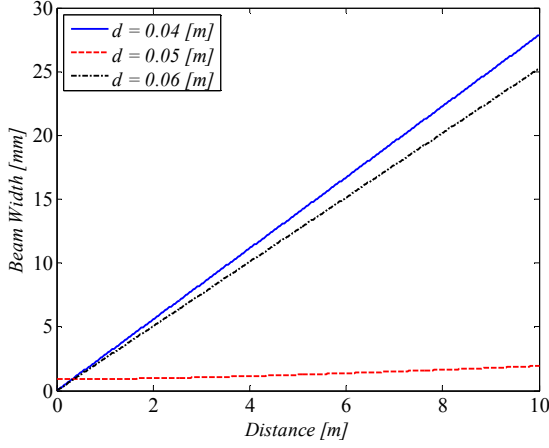


Figure 6: Beam width of reflected beam versus the distance between the mirror and the beam for incident angle of 45° and different d s. $d = 0.04 \text{ m}$ solid line, $d = 0.05 \text{ m}$ dashed line, and $d = 0.06 \text{ m}$ dash-dotted line.

7. CONCLUSION

The ABCD matrices of reflected and refracted Gaussian laser beams impinging a tilted elliptic concave/convex paraboloid surface have been

$$\begin{aligned}
 \mathbf{M}_S^r &= \begin{bmatrix} 1 & 0 \\ -\frac{1}{a}[\cos \theta_0 + \cos(2\phi_i + \theta_0)] & 1 \end{bmatrix} \\
 \mathbf{M}_T^r &= \begin{bmatrix} 1 & 0 \\ -\frac{1}{b} \times \frac{\cos^2(\theta_0 + \phi_i)}{\cos^2 \phi_i} [\cos \theta_0 + \cos(2\phi_i + \theta_0)] & 1 \end{bmatrix} \\
 \mathbf{M}_S^t &= \begin{bmatrix} 1 & 0 \\ -\frac{1}{na}(\cos \theta_0 - \cos(\phi_i + \theta_0)\sqrt{n^2 - \sin^2 \phi_i} - \sin(\phi_i + \theta_0)\sin \phi_i) & \frac{1}{n} \end{bmatrix} \\
 \mathbf{M}_T^t &= \begin{bmatrix} \frac{\sqrt{n^2 - \sin^2 \phi_i}}{n \cos \phi_i} & 0 \\ -\frac{1}{b} \times \frac{\cos \phi_i}{\sqrt{n^2 - \sin^2 \phi_i}}(\cos \theta_0 - \cos(\phi_i + \theta_0)\sqrt{n^2 - \sin^2 \phi_i} - \sin(\phi_i + \theta_0)\sin \phi_i) & \frac{n^2 \cos^3 \phi_i}{(n^2 - \sin^2 \phi_i)^{3/2}} \end{bmatrix} \\
 \mathbf{M}_S^b &= \begin{bmatrix} 1 & 0 \\ -\frac{(1 - n^2)(\cos \theta_0 - n \sin \theta_0)}{n(1 + n^2)a} & \frac{1}{n} \end{bmatrix} \\
 \mathbf{M}_T^b &= \begin{bmatrix} n & 0 \\ -\frac{(1 - n^2)(\cos \theta_0 - n \sin \theta_0)}{n^2(1 + n^2)b} & \frac{1}{n^2} \end{bmatrix}
 \end{aligned}$$

Note that above matrices are valid for concave paraboloid where $\theta_0 < \theta_n$. One can obtain concave paraboloid for $\theta_0 > \theta_n$ by changing θ_0 into $-\theta_0$. To change the matrices into convex paraboloid case θ_0 , a , and b must change into $-\theta_0$, $-a$, and $-b$, respectively.

REFERENCES

- [1] H. Kogelnik and T. Li, "Laser beams and resonators," Appl. Opt., vol. 5, pp. 1550-1567, 1966.
- [2] M. Shabani and A. A. Shishegar, "Vectorial Gaussian beam expansion for high-frequency wave propagation," IET Microwaves, Antennas & Propagation, vol. 4, pp. 2014-2023, 2010.

obtained. The method of obtaining matrixes includes simplification level, converting all coordinates into a unique reference coordinates and applying electromagnetic field boundary condition on complex phase of the Gaussian beam. The method was applied to paraboloid surfaces in this text. However, by choosing appropriate transformation, it would be possible to formulate ABCD matrix for other types of interfaces as well. To show the application of such equations, a practical case which consisted of a parabola of revolution mirror reflecting a laser located on mirror principle axis, was studied. It was shown that sagittal and tangential planes have the same imaging focal length and a parabola of revolution mirror has the same focal length for different angle of tilting.

Appendix I

In this appendix each ABCD matrix is represented by $\mathbf{M}_v^u$, where u can be r , t , and b referring to reflected, refracted, and refracted at Brewster angle beams. The subscript v refers to S , or T regarding the sagittal or tangential plane.

- [3] C. Qi, X. Shi, and G. Wang, "High-order circuit-level thermal model of vertical-cavity surface-emitting lasers," IET Optoelectronics, vol. 5, pp. 19-27, 2011.
- [4] Z. Zhao, K. Duan, and B. Lü, "Non-equiphase Hermite-Gaussian model of diodelaser beams," Optik - International Journal for Light and Electron Optics, vol. 119, pp. 167-170, 2008.
- [5] J. H. Song, "Fibre coupling tolerance modelling of uniform grating coupler on silicon on insulator," Electronics Letters, vol. 47, pp. 1290-1292, 2011.
- [6] A. Chabory, J. r. m. Sokoloff, S. Bolioli, and P. F. o. Combes, "Computation of electromagnetic scattering by multilayer dielectric objects using Gaussian beam based techniques," Comptes Rendus Physique, vol. 6, pp. 654-662, 2005/8/ 2005.
- [7] A. Chabory, J. Sokoloff, and S. Bolioli, "Physically based expansion on conformal Gaussian beams for the radiation of

- curved aperture in dimension 2," IET Microwaves, Antennas & Propagation, vol. 2, pp. 152-157, 2008.
- [8] J. S. Gardner, "Approximate expansion of a narrow Gaussian beam in spherical vector wave functions," Antennas and Propagation, IEEE Transactions on, vol. 55, pp. 3172-3177, 2007.
 - [9] I. Toselli, L. C. Andrews, R. L. Phillips, and V. Ferrero, "Free space optical system performance for a Gaussian beam propagating through non-Kolmogorov weak turbulence," Antennas and Propagation, IEEE Transactions on, vol. 57, pp. 1783-1788, 2009.
 - [10] H. Mao and D. Zhao, "Intensity distribution and coherence property for the broadband Gaussian Schell-model array beams in free space," Optics Communications, vol. 284, pp. 3795-3801, 2011.
 - [11] R. Shavit, J. Sangiolo, and T. Monk, "Scattering analysis of arbitrarily shaped cylinders in a focused beam system-oblique incidence case," IEE Proceedings - Microwaves, Antennas and Propagation, vol. 148, pp. 73-78, 2001.
 - [12] W. Zhen-Sen, L. Zheng-Jun, L. Huan, Y. Qiong-Kun, and L. Hai-Ying, "Off-axis Gaussian beam scattering by an anisotropic coated sphere," Antennas and Propagation, IEEE Transactions on, vol. 59, pp. 4740-4748, 2011.
 - [13] C. Rieckmann, M. R. Rayner, and C. G. Parini, "Diffracted Gaussian beam analysis of quasi-optical multi-reflector systems," Electronics Letters, vol. 36, pp. 1600-1601, 2000.
 - [14] H. T. Chou and P. H. Pathak, "Fast Gaussian beam based synthesis of shaped reflector antennas for contoured beam applications," Microwaves, Antennas and Propagation, IEE Proceedings, vol. 151, pp. 13-20, 2004.
 - [15] D. Luga, D. Luga, A. Boag, and C. Letrou, "Gaussian beam tracking through a curved interface: comparison with a method of moments," IEE Proceedings - Microwaves, Antennas and Propagation, vol. 150, pp. 49-55, 2003.
 - [16] H. Liu, L. Liu, R. Xu, and Z. Luan, "ABCD matrix for reflection and refraction of Gaussian beams at the surface of a parabola of revolution," Appl. Opt., vol. 44, pp. 4809-4813, 2005.
 - [17] Y. Yu and W. Dou, "ABCD matrix for reflection and refraction of Gaussian beams on the interface of an elliptic paraboloid," Journal of Infrared, Millimeter, and Terahertz Waves, vol. 31, pp. 1304-1311, 2010.
 - [18] G. A. Massey and A. E. Siegman, "Reflection and refraction of Gaussian light beams at tilted ellipsoidal surfaces," Appl. Opt., vol. 8, pp. 975-978, 1969.
 - [19] S. Gangopadhyay and S. Sarkar, "ABCD matrix for reflection and refraction of Gaussian light beams at surfaces of hyperboloid of revolution and efficiency computation for laser diode to single-mode fiber coupling by way of a hyperbolic lens on the fiber tip," Appl. Opt., vol. 36, pp. 8582-8586, 1997.
 - [20] T. J. Finn, N. Trappe, J. A. Murphy, and S. Withington, "The Gaussian beam mode analysis of off-axis aberrations in long wavelength optical systems," Infrared Physics Technology, vol. 51, pp. 351-359, 2008.
 - [21] A. W. M. Lee, Q. Qin, S. Kumar, B. S. Williams, Q. Hu, and J. L. Reno, "Real-time terahertz imaging over a standoff distance (> 25 meters)," Applied Physics Letters, vol. 89, pp. 141125-3, 2006.
 - [22] X. Wang, Y. Cui, D. Hu, W. Sun, J. Ye, and Y. Zhang, "Terahertz quasi-near-field real-time imaging," Optics Communications, vol. 282, pp. 4683-4687, 2009.
 - [23] R. Yano, H. Gotoh, Y. Hirayama, T. Hattori, and S. Miyashita, "Synthesis of terahertz electromagnetic wave pulses using amplitude-and-phase masks," Chemical Physics, vol. 326, pp. 577-582, 2006.
 - [24] B. E. A. Saleh and M. C. Teich, Fundamentals of photonics. New York, Chichester: Wiley, 1991.
 - [25] D. E. Goldberg, Genetic algorithms in search, optimization, and machine learning. Reading, Mass: Addison-Wesley, 1989.
 - [26] J. F. Bonnans, Numerical optimization: theoretical and practical aspects, 2nd ed. ed. Berlin, New York: Springer, 2006.
 - [27] B. Stephen and V. Lieven, Convex Optimization: Cambridge University Press, 2004.

BIOGRAPHIES



Mojtaba Mansour Abadi was born in Fasa, Iran, in 1982. He received his B.Sc. degree in Electrical Engineering from Islamic Azad University, Fasa, Iran in 2005, and he was awarded M.Sc. in Electromagnetic Fields and Waves from K.N. Toosi University of Technology, Tehran, Iran in 2008. In 2012 he joined Optical Communications Research Group, Northumbria University, Newcastle, UK to pursue a Ph.D. degree in Optical Communication funded by the University. His research is focused on free space optical (FSO) communication and hybrid FSO/RF systems.



Zabih Ghassemloo, CEng, Fellow of IET, Senior Member of IEEE Received his B.Sc. (Hons) degree in Electrical and Electronics Engineering from the Manchester Metropolitan University in 1981, and his M.Sc. and Ph.D. in Optical Communications from the University of Manchester Institute of Science and Technology (UMIST) with scholarships from the Engineering Research Council (UK), in 1984 and 1987, respectively. From 1986-87 worked in UMIST and from 1987 to 1988 was a Post-doctoral Research Fellow at the City University, London. 1988 he joined Sheffield Hallam University as a Lecturer, becoming a Reader in 1995 and a Professor in Optical Communications in 1997. 2004 - 2012 was an Associate Dean for Research in the School of Computing, Engineering, 2012-2014 was an Associate Dean for Research and Innovation in the Faculty of Engineering and Environment, at Northumbria University at Newcastle, UK. He currently is heads of the Northumbria Communications Research Laboratories within the Faculty. 2001 he was a recipient of the Tan Chin Tuan Fellowship in Engineering from the Nanyang Technological University in Singapore. He has been a visiting professor at a number of institutions and currently is at University Tun Hussein Onn Malaysia, Malaysia. He is the Editor-in-Chief of the International Journal of Optics and Applications, British Journal of Applied Science & Technology, and The Mediterranean Journal Electronics and Communications, also is editor of the IEEE Transactions on Wireless Communications. He also was an associate editor of IEEE Communications Letters and currently is an associate editor the EURASIP Journal of Wireless Communications and Networking, and serves on the editorial board of the International Journal of Communication Systems, Journal of Electrical and Computer Engineering, Iranian Journal Electrical and Electronic Engineering, Contemporary Engineering Sciences, Research Letter in Signal Processing, and Hindawi Journal of Electrical and Computer Engineering. He has been guest editor of a number of special issues in international journals. He is the Vice-Chair of the European Union Cost Action IC1101 OPTIWISE, is a College Member of the Engineering, and Physical Science Research Council, UK, acted as external assessor/examiner for a number of international institutions, and has served on a number of international Research and Advisory Committees including a Panel Member of the Romanian Research Assessment Exercise 2011. His researches interests are on optical wireless and wired communications, visible light communications, free space optics, and radio over fibre/FSO. He has received a number of research grants from UK Research Councils, European Union, Industry and UK Government, has supervised a large number of Ph.D. students (more than 48) and published over 550 papers (196 journals + 4 books) and presented a large number of keynote and invited talks (around 70). He is a co-author of a CRC book on "Optical Wireless Communications - Systems and Channel Modelling with Matlab (2012); a co-editor of an IET book on "Analogue Optical Fibre Communications". He is the founder and the Chairman of the IEEE, IET International Symposium on Communication Systems, Network and Digital Signal Processing, the Chairman of 16th European Conference on Networks and Optical Communications 2011, and EU Cost Action IC1101 2nd Workshop on Optical Wireless

Communications 2013. He is organizer/co-organizer of a number of international colloquiums/workshops etc and is a member of the steering and technical committees of a number of international conferences. From 2004-06 he was the IEEE UK/IR Communications Chapter Secretary, the Vice-Chairman (2004-2008), the Chairman (2008-2011), and Chairman of the IET Northumbria Network (Oct 2011-). Personal Web site: <http://soe.northumbria.ac.uk/ocr/people/ghassemlooy/>



Wai Pang Ng received his BEng (Communication and Electronic Engineering) from Northumbria University in Newcastle upon Tyne, UK in 1997 and Ph.D. (Electrical Engineering) from University of Wales at Swansea, UK in 2001. He joined Northumbria University as a Lecturer in 2014 and currently is a Reader (Associate Professor) at the Faculty of Engineering and Environment at Northumbria University. He is working within the Optical Communication Research Group. His research interests include optical switching, GMPLS, adaptive digital signal processing and cryptography and has published over 56 papers in international journals and conferences. He is the Chair, IEEE UK&RI Communication Chapter, and Publicity Chair for ICC 2015, London, and was the Co-chair, Signal Processing for Communications Symposium, IEEE ICC 2009.