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Optimal Finite-time Control of Positive Linear Discrete-time Systems

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ABSTRACT

This paper considers solving optimization problem for linear discrete time systems such that closed-loop discrete-time system is positive (i.e., all of its state variables have non-negative values) and also finite-time stable. For this purpose, by considering a quadratic cost function, an optimal controller is designed such that in addition to minimizing the cost function, the positivity property of the optimal state trajectory of the closed-loop system is also guaranteed. Furthermore, state variables of the closed-loop system converge to the origin in finite steps (finite-time stability). In this regard, the positive Linear Quadratic Regulator (LQR+) problem for the linear discrete time systems is stated. Once, the cost function with finite-time horizon is considered and another time the cost function with infinite-time horizon is assumed. In this regard, two theorems are given and proved which consider the problem of building positive and optimize linear time-varying discrete time systems. Results can also be applied to linear time-invariant discrete time systems. Finally, computer simulations are given to illustrate effective performance of the designed controller and also verify the theoretical results.

1. Introduction

Positive systems are kind of systems for which negative region is not defined and if the state variables of the system start from non-negative initial conditions, they will remain non-negative forever. Such systems can be found in different parts of the natural sciences and technology, including biology, chemistry, ecology, economics, sociology and communications [1-3]. Over the past decades, many theoretical issues have been examined for positive systems and it is still continuing. For example, realization, controllability and observability [4-9], input to output or input to state stability [10-12], passivity [13-15], the positive stabilization [16] and optimal and robust control for positive systems [17-21] are some of these issues.

One of the important issues in control theory is optimal control. The goal of optimal control is finding control signals such that in addition to minimizing the certain performance criteria, certain physical constraints are also satisfied [22-27]. Considering nature of positive systems, optimal control theory could play an important key role to get appropriate results. However, optimal control problems are somewhat different for positive systems and in many cases; the positive property of system could not be saved by designing optimal controller that is obtained from solving the standard LQR problem. It is evident that finding the optimizer and building a positive control mechanism in the category of positive linear systems is very important.

Considering major articles within framework of the LQR problem, there are no constraints on state and control input [28, 29], however there are articles that studied the constrained LQR problem. In [30], by using change of the associated controller block, some sufficient conditions are obtained for weighting matrices of square cost function to guarantee positivity of the closed-loop system. Using generalized ideas in this article, [31] was released which has no comprehensive and efficient results for all of the positive systems. Authors in [32] examined minimum energy problem for positive linear time-invariant discrete-time systems with fixed final state. Moreover, authors of [33] studied finite-time horizon LQ+ problem for positive linear systems and obtained necessary and sufficient conditions of optimality using the maximum principle [34]. Sufficient conditions on weighted matrices were given in [35]. These conditions guarantee non-negativity of state variables only in special cases. In [36], a solution was achieved for the LQR problem with finite-time horizon for linear time-invariant systems. According to this study, goals of the optimal control is achieved and state variables are remained non-negative by selecting some especial initial conditions.

Therefore, the literature survey show that the positive LQR problem with non-negativity constraint on state variables is still unresolved for general cases. This paper studies optimal finite-time control problem for linear time-varying discrete-time systems with non-negativity constraint on state variables for closed-loop system such that the state-variables converge to the origin in a finite-time. The proposed approach is introduced for both finite-time horizon and infinite-time horizon cost functions, analytically. Furthermore, achieved results are applied to linear time-invariant discrete-time examples.

The remainder of this paper is as follows: in the next section, some basic definitions are given. In the third section, problem formulation is given. The fourth section contains main results of this paper and the optimal control law is designed in this section. In this regard, some theorems are provided. Simulations are given in Section 5. Finally, section 6 gives some concluding remarks.

2. Basic Definitions

Symbol $\Re$ presents real numbers and symbols $\Re^n$ and $\Re^{n\times n}$ illustrate space of column vectors of size n with real entries and space of $n\times n$ matrices with real entries, respectively. For $x\in\Re^n$ and $i=1,\dots,n$, x_i denotes i^{th} component of x .For $A\in\Re^{n\times n}$, a_{ij} denotes $(i,j)^{th}$ entry of A .

Let define:

$$\Re_+ := \{x\in\Re : x\geq 0\}$$

$$\Re_+^n := \{x\in\Re^n : x_i\geq 0,\ 1\leq i\leq n\}$$

For $x\in\Re^n$ and $1\leq i\leq n$, we have:

$$x\geq 0 \quad if \quad x_i\geq 0$$

$$x>0 \quad if \quad x_i>0$$

In this paper, concepts of *positive definite* (*pd*) and *positive semi-definite* (*psd*) will be displayed by the following symbols and the following relationships are dominant:

$$R\succ 0 \;\Rightarrow\; R \;\; is \;\; pd$$

$$Q,S\succeq 0 \;\Rightarrow\; Q,S \;\; is \;\; psd$$

Definition 1. Time-invariant matrix M is non-negative if and only if all of its entries have non-negative values [1].

Definition 2. Time-varying matrix $M[k]$ is non-negative on time interval $[i,N]$ if and only if all of its entries have non-negative values for every $k\in[i,N]$ [1].

In positive time-invariant systems, state variables remained non-negative for all times. However, positivity of time-varying systems is defined on time interval and if the state variables of the time-varying system remained non-negative on the defined time-interval, then the system is positive on the defined time interval [1, 37].

Definition 3. The following discrete-time system

$$\begin{aligned} x[k+1]&=A[k]x[k]+B[k]u[k] \\ y[k]&=C[k]x[k] \end{aligned} \quad ,k\in[i,N] \tag{1}$$

is positive on time interval $[i,N]$ if for any non-negative initial condition, the states and the outputs of system (1) remain non-negative on the time intrval $[i,N]$ [37]. In other words, one has:

$$\begin{aligned} x[k]&\geq 0 \\ y[k]&\geq 0 \end{aligned} \quad for \quad all \quad k\in[i,N],\ x[0]\geq 0 \tag{2}$$

Definition 4. The discrete-time system (1) is finite-time stabilizable if there exist the controller $u[k]$ such in the closed-loop system $x[k]=0$ for all $k\geq k_s$, where $k\ \&\ k_s\in Z^+$ and Z^+ is the set of integer numbers. Also, k_s is the settling time [38].

3. Problem Formulation

Consider the following state space equations of a linear time-varying discrete-time system:

$$x[k+1]=A[k]x[k]+B[k]u[k], k\in[i,N], u\in\Re^m, x\in\Re_+^n, x[i]\geq 0 \tag{3}$$

Cost function has the quadratic form with finite-time horizon as follows:

$$J_i=\frac{1}{2}x^T[N]S[N]x[N]+\frac{1}{2}\sum_{k=i}^{N-1}\left(x^T[k]Q[K]x[k]+u^T[k]R[K]u[k]\right) \tag{4}$$

Purpose of optimization is to determine the control law $u^*[k]$ such that in addition to minimizing the cost function, the positivity of the closed-loop system is ensured. This problem is called discrete-time LQR_N^+ problem.

Furthermore, if the cost function has the following structure:

$$J_0 = \frac{1}{2}\sum_{k=0}^{\infty}(x^T[k]Qx[k] + u^T[k]Ru[k]) \tag{5}$$

with positivity constraint of the closed-loop system, then the optimal control problem is called the discrete-time LQR_∞^+ problem.

As previously stated, state variables of the positive systems just should remain in the non-negative region. When the standard optimal LQR controller is designed for a positive system, there is no guarantee that the closed-loop system is remained positive and in most cases, the state variables of the closed-loop system may enter into the negative region. In this case, k_{en-i} and k_{ex-i} are called an *entry steps* to the negative region and an *exit steps* from the negative region, respectively. Figure (1) shows this issue, clearly. In this figure, $(k_{en}1)$ is the first time step that the state variable enter into the negative region (entry step) and $(k_{ex}1)$ is the first step that the state variable exists the negative region (exist step).

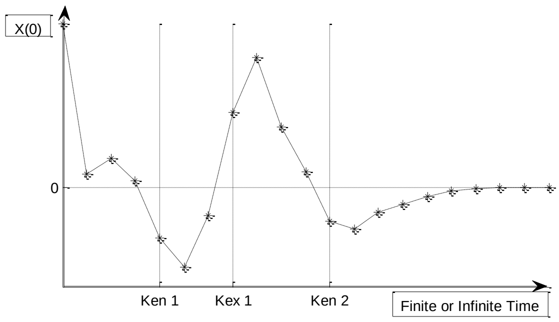

Figure 1: Time-response of state variable via applying the standard LQR problem on a positive system.

In this paper, the purpose is design a controller such that in addition to optimizing the system; it prevents entrance of state variables of the closed-loop system to the negative region and guarantees reaching to the origin in finite steps (i.e., finite-time stability).

4. The Main Results

The proposed idea in this paper is based on determination of the *entry step* k_{en}. In fact, if control law which is obtained from solving the standard LQR problem is applied to the given discrete-time system, k_{en} is the first step which at least one of the state variables of the system has entered to the negative region. This step is determined by applying the obtained control law from solving the standard LQR problem for the given system. This control law should be corrected so that the state variables of the closed-loop system are not allowed to enter into the negative region. Therefore, by adding a sentence to the obtained control law of the standard LQR problem in step $k_{en}-1$, all of the state variables will be equal to zero in finite step (k_{en}).

A. Solving the discrete-time LQR_N^+ Problem

The optimal finite-time controller is given in the following theorem to solve the LQR_N^+ problem:

Theorem 1. Consider state space equations of positive linear time-varying system (3) with cost function (4). Assuming $B[k] \geq 0$, the following control law is the optimal finite-time controller related to the considered LQR_N^+ problem:

$$u[k] = \begin{cases} -K[k]x[k] & k = i, \ldots, k_{en}-2 \\ -K[k]x[k] - H_0V_0 & k = k_{en}-1 \\ 0 & k = k_{en}, \ldots, N-1 \end{cases} \tag{6}$$

In addition, state-space equations of the closed-loop system are as follows:

$$x[k+1] = \begin{cases} \left(A[k] - B[k]K[k]\right)x[k] & k = i, \ldots, k_{en}-2 \\ 0 & k = k_{en}-1, \ldots, N-1 \end{cases} \tag{7}$$

where k_{en} is the first *entry step* that the state variables of system $x[k+1] = \left(A[k] - B[k]K[k]\right)x[k]$ are entered to the negative region. Moreover, $K[k]$, H_0 and V_0 are obtained using the following relations:

$$\begin{aligned} S[k] = {} & A^T[k](S[k+1] - S[k+1]B[k]\left(B^T[k]S[k+1]B[k] + R[k]\right)^{-1} \\ & \times B^T[k]S[k+1])A[k] + Q[k]; \qquad k < N,\ S_N \text{ is given} \end{aligned} \tag{8}$$

$$\begin{aligned} K[k] = {} & \left(B^T[k]S[k+1]B[k] + R[k]\right)^{-1} \\ & \times B^T[k]S[k+1]A[k] \end{aligned} \tag{9}$$

$$\begin{aligned} H_0 = {} & \left(B^T[k_{en}-1]S[k_{en}]B[k_{en}-1] + R[k_{en}-1]\right)^{-1} \\ & \times B^T[k_{en}-1] \end{aligned} \tag{10}$$

Proof: Step (1): Organize the following Hamiltonian function:

$$\begin{aligned} H[k] = {} & \frac{1}{2}\left(x^T[k]Q[k]x[k] + u^T[k]R[k]u[k]\right) \\ & + \lambda^T[k+1]\left(A[k]x[k] + B[k]u[k]\right) \end{aligned} \tag{11}$$

Step (2): Obtain the state equations, the co-states equations and the stationary condition:

$$\begin{aligned} x[k+1] &= \frac{\partial H[k]}{\partial \lambda[k+1]} \\ &= A[k]x[k] + B[k]u[k] \end{aligned} \tag{12}$$

$$\lambda[k]=\frac{\partial H[k]}{\partial x[k]}=Q[k]x[k]+A^T[k]\lambda[k+1]$$
$$0=\frac{\partial H[k]}{\partial u[k]}=R[k]u[k]+B^T[k]\lambda[k] \quad (13)$$

$$u[k]=-R^{-1}[k]B^T[k]\lambda[k+1] \quad (14)$$

Step (3): According to Sweep method [39] and by adding an additional term $V[k]$ to the Lagrange multiplier; consider $\lambda[k]$ the as follows:

$$\lambda[k]=\begin{cases} S[k]x[k] & k=i,\dots,k_{en}-2 \\ S[k]x[k]+V[k] & k=k_{en}-1 \\ S[k]x[k] & k=k_{en},\dots,N-1 \end{cases} \quad (15)$$

By inserting (14) and (15) into relation (12), one has:

$$x[k+1]=\begin{cases} \left(I+B[k]R^{-1}[k]B^T[k]S[k+1]\right)^{-1}A[k]x[k]; & k=i,\dots,k_{en}-2 \\ \left(I+B[k]R^{-1}[k]B^T[k]S[k+1]\right)^{-1} \\ \quad\times\left(A[k]x[k]-B[k]R^{-1}[k]B^T[k]V[k+1]\right); & k=k_{en}-1 \\ \left(I+B[k]R^{-1}[k]B^T[k]S[k+1]\right)^{-1}A[k]x[k]; & k=k_{en},\dots,N-1 \end{cases} \quad (16)$$

Furthermore, putting (15) and (16) into (14) results in:

$$u[k]=\begin{cases} -\left(B^T[k]S[k+1]B[k]+R[k]\right)^{-1}B^T[k] \\ \quad\times S[k+1]A[k]x[k]; & k=i,\dots,k_{en}-2 \\ -\left(B^T[k]S[k+1]B[k]+R[k]\right)^{-1}B^T[k] \\ \quad\times\left(S[k+1]A[k]x[k]+V[k+1]\right); & k=k_{en}-1 \\ -\left(B^T[k]S[k+1]B[k]+R[k]\right)^{-1}B^T[k] \\ \quad\times S[k+1]A[k]x[k]; & k=k_{en},\dots,N-1 \end{cases} \quad (17)$$

By considering the following relations, we can rewrite the control law in a simpler form:

$$\begin{cases} K[k]=\left(B^T[k]S[k+1]B[k]+R[k]\right)^{-1}B^T[k]S[k+1]A[k] \\ H_0=\left(B^T[k_{en}-1]S[k_{en}]B[k_{en}-1]+R[k_{en}-1]\right)^{-1}B^T[k_{en}-1], \\ V[k_{en}]=V_0 \end{cases}$$
$$\Rightarrow u[k]=\begin{cases} -K[k]x[k] & k=i,\dots,k_{en}-2 \\ -K[k]x[k]-H_0V_0 & k=k_{en}-1 \\ -K[k]x[k] & k=k_{en},\dots,N-1 \end{cases} \quad (18)$$

Step (4): Using equations (13), (15) and (16), one has:

$$S[k]x[k]=\begin{cases} Q[k]x[k]+A^T[k]S[k+1]x[k+1]; & k=i,\dots,k_{en}-2 \\ Q[k]x[k]+A^T[k](S[k+1]x[k+1] \\ \quad +V[k+1])-V[k]; & k=k_{en}-1 \\ Q[k]x[k]+A^T[k]S[k+1]x[k+1]; & k=k_{en},\dots,N-1 \end{cases}$$

$$(S[k]-A^T[k]S[k+1]\left(I+B[k]R^{-1}[k]B^T[k]S[k+1]\right)^{-1}$$
$$\times A[k]-Q[k])x[k]=0;\ k=i,\dots,N-1\ \&\ k\neq k_{en}-1$$

$$(S[k]-A^T[k]S[k+1]\left(I+B[k]R^{-1}[k]B^T[k]S[k+1]\right)^{-1}$$
$$\times A[k]-Q[k])x[k]=-V[k]+(A^T[k]-A^T[k]S[k+1]$$
$$\times\left(I+B[k]R^{-1}[k]B^T[k]S[k+1]\right)^{-1}B[k]R_k^{-1}B_k^T)V[k+1];\ k=k_{en}-1 \quad (19)$$

Now, using the matrix inversion lemma [38], the following result can be achieved:

$$\left(I+B[k]R^{-1}[k]B^T[k]S[k+1]\right)^{-1}=$$
$$I-B[k]\left(B^T[k]S[k+1]B[k]+R[k]\right)^{-1}B^T[k]S[k+1] \quad (20)$$

Using the equations (19) and (20) and putting both sides equal to zero in equation (19), one has:

$$S[k]=A^T[k](S[k+1]-S[k+1]B[k]\left(B^T[k]S[k+1]B[k]+R[k]\right)^{-1}$$
$$\times B^T[k]S[k+1])A[k]+Q[k] \quad (21)$$

$$V[k]=\left(A[k]-B[k]K[k]\right)^TV[k+1] \quad (22)$$

For solving the equation (21), the boundary condition $S[N]$ is needed which is given in the cost function [4]. In this case, the sequences $S[k]$ and $K[k]$ are obtained. Also, $V[k_{en}]=V_0$ is chosen such that all of state variables reach to the origin in the *entry step* k_{en}, and thus the finite-time stabilization is also guaranteed . In the other words, we have:

$$x[k_{en}]=\left(A[k_{en}-1]-B[k_{en}-1]K[k_{en}-1]\right)$$
$$\times x[k_{en}-1]-B[k_{en}-1]H_0V_0=0$$
$$\Rightarrow V_0=\left(B[k_{en}-1]H_0\right)^{-1}(A[k_{en}-1]-B[k_{en}-1]$$
$$\times K[k_{en}-1])x[k_{en}-1] \quad (23)$$

Since $x[k_{en}]=0$, by applying $u[k]=0$ for $k\geq k_{en}$ and considering $x[k+1]=A[k]x[k]+B[k]u[k]$, it results in $x[k]=0$ for $k\geq k_{en}$ and the finite-time convergence of the state-variables to the origin (in finite steps) is achieved. Also, considering this point the control laws (18) and (6) are similar. ■

B. Solving the discrete-time LQR_∞^+ Problem

The LQR_∞^+ problem is given in the following theorem:

Theorem 2. Consider the following state space equations of the positive linear time-invariant system (24) and the cost function (5).

$$x[k+1]=Ax[k]+Bu[k],\ k>0, u\in\Re^m, x\in\Re_+^n, x[i]\geq 0 \tag{24}$$

Assuming $B\geq 0$, the LQR_∞^+ problem has the following solution:

$$u[k]=\begin{cases}-K[\infty]x[k] & k=i,\dots,k_{en}-2\\ -K[\infty]x[k]-H_0V_0 & k=k_{en}-1\\ 0 & k=k_{en},\dots,\infty\end{cases} \tag{25}$$

and the state-space equations of the closed-loop system are as follows:

$$x[k+1]=\begin{cases}(A-BK[\infty])x[k] & k=i,\dots,k_{en}-2\\ 0 & k=k_{en}-1,\dots,\infty\end{cases} \tag{26}$$

where

$$S[\infty]=A^T\left[S[\infty]-S[\infty]B\left(B^TS[\infty]B+R\right)^{-1}B^TS[\infty]\right]A+Q \tag{27}$$

$$K[\infty]=\left(B^TS[\infty]B+R\right)^{-1}B^TS[\infty]A \tag{28}$$

$$H_0=\left(B^TS[\infty]B+R\right)^{-1}B^T \tag{29}$$

$$V_0=(BH_0)^{-1}(A-BK[\infty])x[k_{en}-1] \tag{30}$$

Also, k_{en} was introduced before in the theorem (1).

Proof: If we write the results of the theorem (1) for time-invariant systems, relations (8), (9), (10) and (11) are changed as follows:

$$S[k]=A^T(S[k+1]-S[k+1]B\left(B^TS[k+1]B+R\right)^{-1}\times B^TS[k+1])A+Q \tag{31}$$

$$K[k]=\left(B^TS[k+1]B+R\right)^{-1}B^TS[k+1]A \tag{32}$$

$$H_0=\left(B^TS[k_{en}]B+R\right)^{-1}B^T \tag{33}$$

$$V_0=(BH_0)^{-1}(A-BK[k])x[k_{en}-1] \tag{34}$$

In [39], it was proved that if the couple $\{A,B\}$ is stabilizable, then there is a unique positive-defined answer (i.e., $S[\infty]$) for the equation (31) where k tends to infinity. Therefore, the time-varying control gain $K[k]$, will changed to the time-invariant control gain $K[\infty]$ and the relations (31), (32), (33) and (34) will be changed to the relations (27), (28), (29) and (30), respectively.

5. Simulation

In this section, a numerical example is given to verify the theoretical results.

Example: Consider the following positive system (35) with the given cost function (36):

$$x[K+1]=Ax[K]+Bu[K],\quad A=\begin{bmatrix}0.9 & 0.1\\ 0.6 & 0.5\end{bmatrix}, B=\begin{bmatrix}0.9\\ 0.8\end{bmatrix}, x[0]=\begin{bmatrix}4\\ 2\end{bmatrix} \tag{35}$$

$$J_0=\frac{1}{2}\sum_{k=0}^{\infty}\left(x^T[K]Qx[k]+u^T[k]Ru[k]\right), Q=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, R=1 \tag{36}$$

A. The zero input solution of system

Eigenvalues of the open-loop system are as follows:

$$|z\,\mathrm{I}_2-A|=0\Rightarrow z_1=0.3838, z_2=1.0162 \tag{37}$$

Since z_2 is bigger than one, the open-loop system is unstable. Figure (2) shows time history of the state variables x for the open-loop system (35) which is a positive system. As seen, zero input solution of system is unstable.

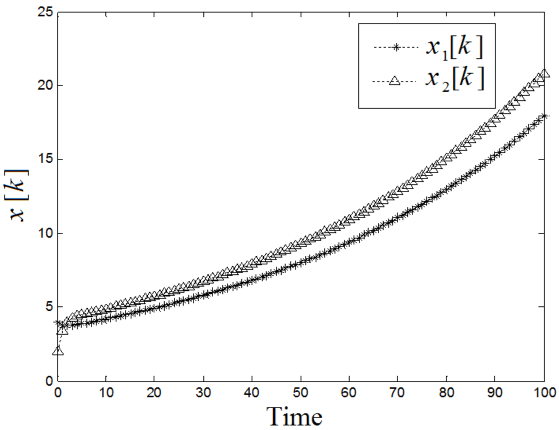

Figure 2: Time history of state variables of the open-loop system.

B. Solving the standard LQR problem

With solving discrete-time algebraic Riccati equation (ARE) by using MATLAB software, we have:

$$S[\infty]=\begin{bmatrix}1.5931 & 0.1366\\ 0.1366 & 1.1785\end{bmatrix} \tag{38}$$

$$K[\infty]=\begin{bmatrix}0.6257 & 0.2120\end{bmatrix} \tag{39}$$

The results of applying the standard LQR problem are as follows:

$$x[k+1]=\begin{bmatrix}0.3368 & -0.0908\\ 0.0994 & 0.3304\end{bmatrix}x[k] \qquad (40)$$

$$x[0]=\begin{bmatrix}4\\2\end{bmatrix},\ x[1]=\begin{bmatrix}1.166\\1.058\end{bmatrix},\ x[2]=\begin{bmatrix}0.2966\\0.4656\end{bmatrix}$$
$$x[3]=\begin{bmatrix}0.0576\\0.1833\end{bmatrix},\ x[4]=\begin{bmatrix}0.0027\\0.0663\end{bmatrix},\ x[5]=\begin{bmatrix}-0.005\\0.0222\end{bmatrix}$$
$$x[6]=\begin{bmatrix}-0.003\\0.0068\end{bmatrix},\ x[7]=\begin{bmatrix}-0.0018\\0.0019\end{bmatrix},\ x[8]=\begin{bmatrix}-0.0008\\0.0004\end{bmatrix} \qquad (41)$$

$$\left|z\,I_2-\left(A-BK[\infty]\right)\right|=0\Rightarrow\begin{cases}z_1=0.3336+0.0950i\\ z_2=0.3336-0.0950i\end{cases} \qquad (42)$$

$$u_{LQR}[k]=-0.6257x_1[k]-0.2120x_2[k] \qquad (43)$$

$$\begin{aligned}&u_{LQR}[0]=-2.927,\ u_{LQR}[1]=-0.9539,\\ &u_{LQR}[2]=-0.2843,\ u_{LQR}[3]=-0.0749,\\ &u_{LQR}[4]=-0.0158,\ u_{LQR}[5]=-0.0015\\ &u_{LQR}[6]=0.0009,\ u_{LQR}[7]=0.0007\end{aligned} \qquad (44)$$

By attention to the value of state variables in (40), it reveals that the closed-loop system (40) (by controller (43)) is asymptotically stable and travels to the optimal path. However, with paying attention to the value of the first element of the state vector in steps 5 to 8, it can be determined that the closed-loop system does not remain positive. Therefore, the controller (43) does not guarantee the positivity of the closed-loop system.

C. Designing optimal finite-time LQR_{∞}^{+} controller

Regarding to the values of the state variables in the equation (41) we have:

$$x[4]=\begin{bmatrix}0.0027\\0.0663\end{bmatrix},x[5]=\begin{bmatrix}-0.005\\0.0222\end{bmatrix}\Rightarrow K_{en}=5 \qquad (45)$$

Therefore, according to Theorem (2), one has:

$$u[k]=\begin{cases}-0.6257x_1[k]-0.2120x_2[k] & k=0,\dots,3\\ -0.6257x_1[k]-0.2120x_2[k]-0.0313 & k=4\\ 0 & k=5,\dots,\infty\end{cases} \qquad (46)$$

$$\begin{aligned}&u_{LQR^+}[0]=-2.927,\ u_{LQR^+}[1]=-0.9539\\ &u_{LQR^+}[2]=-0.2843,\ u_{LQR^+}[3]=-0.0749\\ &u_{LQR^+}[4]=-0.0471,\\ &u_{LQR^+}[5]=u_{LQR^+}[6]=u_{LQR^+}[7]=0\end{aligned} \qquad (47)$$

$$x[k+1]=\begin{cases}\left(A-BK[\infty]\right)x[k]\\ 0\end{cases} \qquad (48)$$

$$x[0]=\begin{bmatrix}4\\2\end{bmatrix},\ x[1]=\begin{bmatrix}1.166\\1.058\end{bmatrix},\ x[2]=\begin{bmatrix}0.2966\\0.4656\end{bmatrix} \qquad (49)$$
$$x[3]=\begin{bmatrix}0.0576\\0.1833\end{bmatrix},\ x[4]=\begin{bmatrix}0.0027\\0.0663\end{bmatrix},\ x[5]=\begin{bmatrix}0\\0\end{bmatrix}$$
$$x[6]=\begin{bmatrix}0\\0\end{bmatrix},\ x[7]=\begin{bmatrix}0\\0\end{bmatrix},\ x[8]=\begin{bmatrix}0\\0\end{bmatrix}$$

The system (48), is finite-time stable ($x[k]=0$ for $k\geq 5$) and travels the optimal path with high convergence speed. Figure (3), shows the optimal trajectory of state variables of the closed-loop system (35) with the optimal LQR_{∞}^{+} controller (46). As seen, the positivity of the closed-loop system is achieved and also the closed-loop system is finite-time stable.

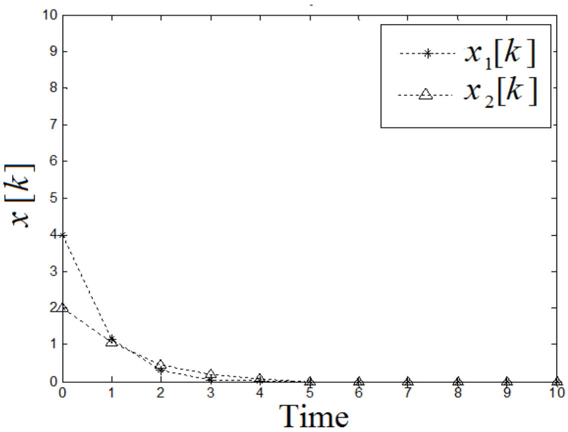

Figure 3: Time history of the state variables for the closed-loop system with LQR_{+}^{∞} controller.

6. CONCLUSION

In this article, the finite-time LQR+ (positive LQR) problem for linear discrete-time systems was expressed and it was solved for cost functions with finite-time horizon and infinite-time horizon. In this regard, two theorems were given to design the optimal controller, which guarantee positivity of the closed-loop system and its finite-time stabilization. A numerical example was also given to show accuracy and efficacy of the achieved results.

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