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Research paper

Improved Bilinear Balanced Truncation for Order Reduction of the High-Order Bilinear System Based on Linear Matrix Inequalities

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Abstract

Background and Objectives: This paper proposes a new Model Order Reduction (MOR) method based on the Bilinear Balanced Truncation (BBT) approach. In the BBT method, solving the generalized Lyapunov equations is necessary to determine the bilinear system's controllability and observability Gramians. Since the bilinear systems are generally of high order, the computation of the Gramians of controllability and observability have huge computational volumes. In addition, the accuracy of reduced-order model obtained by BT is relatively low. In fact, the balanced truncation method is only available for local energy bands due to the use of type I Gramians. In this paper, BBT based on type II controllability and observability Gramians would be considered to fix these drawbacks.

Methods: At first, a new iterative method is proposed for determining the proper order for the reduced-order bilinear model, which is related to the number of Hankel singular values of the bilinear system whose real parts are closest to origin and have the most significant amount of energy. Then, the problem of determining of type II controllability and observability Gramians of the high-order bilinear system have been formulated as a constrained optimization problem with some Linear Matrix Inequality (LMI) constraints for an intermediate middle-order system. Then, the achieved Gramians are applied to the BBT method to determine the reduced-order model of the bilinear system. Next, the steady state accuracy of the reduced model would be improved via employing a tuning factor.

Results: Using the concept of type II Gramians and via the proposed method, the accuracy of the proposed bilinear BT method is increased. For validation of the proposed method, three high-order bilinear models are approximated. The achieved results are compared with some well-known MOR approaches such as bilinear BT, bilinear Proper Orthogonal Decomposition (POD) and Bilinear Iterative Rational Krylov subspace Algorithm (BIRKA) methods.

Conclusion: According to the obtained results, the proposed MOR method is superior to classical bilinear MOR methods, but is almost equivalent to BIRKA. It is out-performance respecting to BIRKA is its guaranteed stability and convergence.

Introduction

Bilinear systems are a class of nonlinear systems that serve as a link between linear and nonlinear systems. States and inputs of these systems are linear, but they are jointly nonlinear.

Researchers have been interested in bilinear systems for many years due to several real-world examples exhibiting such behavior. They include power systems [1], heat transfer [2], and electrical circuits [3].

One of the main applications of bilinear systems is the approximation of weakly nonlinear systems with bilinear systems using the Carleman bilinearization [4], [5]. However, the approximation of nonlinear systems via bilinearization methods usually leads to a high-order bilinear model. Analysis, design, and implementation of the high-order bilinear systems are complicated and time-consuming. Therefore, researchers have considered Model Order Reduction (MOR) of bilinear systems for analyzing and control purposes in the literature. Indeed, a reduced-order approximation of the high-order bilinear model is determined to decrease the complexity. For this purpose, MOR methods created for linear systems [6]-[8] were extended to bilinear systems. These methods include proper orthogonal decomposition (POD) [9], moment matching techniques [10], [11], Krylov subspace methods [12]-[14], projection-based methods [15]-[17] and polynomial expansion series based methods [18]-[20].

H_2-optimal MOR methods are the other approaches to approximate the high order systems [21]-[23]. In [23], a special class of linear parameter-varying systems were reformulated as bilinear dynamical systems. Then, a H_2 norm in the generalized frequency domain was minimized based on the gradient descent on the Grassmann manifold. In [24], H_2 optimal MOR problems were investigated for K-power systems as a special class of bilinear systems. Xu et al. shown that the H_2 optimal MOR problem of the bilinear system could be considered as unconstrained minimization problem on Grassmann manifold by using Gramians of controllability and observability [25] and cross Gramians of the bilinear systems [26]. In [27], the Riemannian trust-region method considered this minimization problem on the Stifel manifold. The time-limited and frequency-limited H_2 optimal MOR were other approaches to approximate the high-order bilinear systems in [28], [29].

As one of the most popular MOR methods for linear deterministic control systems, Moore introduced the balanced truncation method in [30]. Hsu et al. in [31] extended the BT method for order reduction of bilinear systems, called bilinear BT or BBT. Afterward, many researchers focused on model order reduction of bilinear systems based on the BT method and improved it [32], [33].

In the BBT method, the Gramians of controllability and observability are crucial. Solving the generalized Lyapunov equations is necessary to determine the bilinear system's controllability and observability Gramians. Since the bilinear systems are generally of high order, the computation of the Gramians of controllability and observability have huge computational volumes. In addition, the accuracy of reduced-order obtained by BT is not clear [34]. In fact, the balanced truncation method is only available for local energy bands due to the use of type I Gramians and no error bounds have been provided for BBT so far [34]. Therefore, high computational volume and relatively low accuracy especially in steady state are major drawbacks of the BBT method [35]-[37]. Despite these drawbacks, the BBT method ensures stability and convergence which makes it suitable for order reduction of the intermediate systems [38]. In order to improve BT for bilinear systems, in [34], BBT was extended based on type II Gramians, where the H_∞ error bounds were achieved for the reduced bilinear system in terms of the truncated Hankel singular values. In computing these type II Gramians, the equality constraints in the generalized Lyapunov equations are replaced with inequality. Therefore, optimal determination of the type II Gramians via these inequalities is essential. However, up to the knowledge of the authors, no solving method has provided for computing these Gramians in [34] and the related literature afterwards, leading to another challenge in this area. Therefore, employing optimal type II Gramians for improving the BBT method accuracy with lower computational volume and preserving BBT benefits would be exciting and essential.

This paper proposes a new method for MOR of bilinear systems based on the BBT method using the LMI approach to increase the accuracy of the BBT method. For this purpose, at first, a new iterative method is proposed for determining the proper order for the reduced-order bilinear model, which is related to the number of Hankel singular values of the bilinear system whose real parts are closest to origin and have the most significant amount of energy. Then, the problem of determining of type II controllability and observability Gramians of the high-order bilinear system have been formulated as a constrained optimization problem with some Linear Matrix Inequality (LMI) constraints for an intermediate middle-order system. Then, the achieved Gramians are applied to the BBT method to determine the reduced-order model of the bilinear system. Next, the steady state accuracy of the reduced model would be improved via employing a tuning factor. The proposed method has the advantage of increasing the accuracy of the BBT method, while its computational complexity is reduced. To evaluate the efficiency of the proposed method, three test systems have been then examined. The achieved results are compared with some well-known MOR methods such as BBT, bilinear Proper Orthogonal Decomposition (POD) and Bilinear Iterative Rational Krylov subspace Algorithm (BIRKA) methods [39]. The results show that the proposed method is superior to classical bilinear MOR methods, but is almost equivalent to BIRKA. It is out-performance respecting BRIKA is its guaranteed stability and convergence. Therefore, the main contribution of the article can be summarized as follows:

- Development of a new iterative method to determine the proper order for the reduced model.
- Introduction of improved BBT method with increased accuracy and reduced computational complexity and steady-state error.
- Reformulation of the generalized Lyapunov equations as an LMI constrained optimization problem for the middle order approximation of system to determine the Gramians of controllability and observability as type II Gramians to reduce computational complexity and improve BBT accuracy.
- Reducing BBT steady-state error by tuning the feedforward gain of the reduced-bilinear model.

The rest of this paper is organized as follows: In section 2, the MOR of bilinear systems is introduced. In section 3, the basics of the bilinear BT method are presented. In section 4, the proposed method for MOR of the bilinear system is introduced.

In section 5, three high-order bilinear test systems are reduced using the proposed MOR method. The achieved results demonstrate that the proposed methods outperform.

Finally, the paper is concluded in section 6.

MOR of Bilinear Systems

Consider the single-input, single-output bilinear system as follows:

$$\zeta:\begin{cases}\dot{x}(t) = Ax(t) + Nx(t)u(t) + Bu(t)\\ y(t) = Cx(t)\end{cases} \tag{1}$$

where, $A, N \in R^{n\times n}, B, C^T \in R^n$ are the matrices of the bilinear system, $x(t) \in R^n$ is the state vector, and $u(t) \in R$ and $y(t) \in R$ are the input and output of the bilinear system, respectively. Also, n is the order of the bilinear system.

Suppose that the bilinear system of (1) is of high order. Model order reduction aims to create a system in which the original bilinear system's and reduced-order approximation's responses are almost identical, i.e., $y(t) \approx y_r(t)$ for all admissible inputs. Further, both (1) and the reduced-order system have the same structure. The reduced-order bilinear model can be represented as follows:

$$\zeta_r:\begin{cases}\dot{x}_r(t) = A_r x_r(t) + N_r x_r(t)u(t) + B_r u(t)\\ y_r(t) = C_r x_r(t)\end{cases} \tag{2}$$

where, $A_r, N_r \in R^{r\times r}, B_r, C_r^T \in R^r$ are the matrices of the reduced-order bilinear system, which are unknown and should be determined, $x_r(t) \in R^r$ is the state vector, and $u(t), y_r(t) \in R$ are the input and output of the reduced-order bilinear system, respectively. Also, $r \ll n$ is the order of the reduced model.

It should be noted that if the original bilinear system is stable, the reduced-order model should be stable, too.

Balanced Truncation for Bilinear Systems

The controllability Gramians of the bilinear system of (1) are defined as follows [35]:

$$P = \sum_{i=1}^{\infty}\int_0^{\infty}\cdots\int_0^{\infty} P_i P_i^T \, dt_1 \cdots dt_i \tag{3}$$

where

$$\begin{aligned}&P_1(t_1) = e^{At_1}B\\ &P_i(t_1,\cdots,t_i) = e^{At_i}NP_{i-1}\end{aligned} \tag{4}$$

Also, the observability Gramians of the bilinear system of (1) is defined as follows:

$$Q = \sum_{i=1}^{\infty}\int_0^{\infty}\cdots\int_0^{\infty} Q_i^T Q_i \, dt_1 \cdots dt_i \tag{5}$$

where

$$\begin{aligned}&Q_1(t_1) = Ce^{At_1}\\ &Q_i(t_1,\cdots,t_i) = Q_{i-1}Ne^{At_i}\end{aligned} \tag{6}$$

Type I Gramians

Theorem 1 [40]. Consider the bilinear system of (1) with a stable matrix *A*. The truncated type I controllability Gramian *P* of the system satisfies the generalized Lyapunov equation given by (7), as:

$$AP + PA^T + NPN^T + BB^T = 0 \tag{7}$$

As a result of extending theorem 1 for observing Gramians, it is concluded that truncated type I Gramians of observability can be obtained by solving the following generalized Lyapunov equation:

$$A^T Q + AQ + N^T QN + C^T C = 0 \tag{8}$$

The following iterative method has been used to solve the generalized Lyapunov equations of (7) and (8) [36].

Initially, the bilinear term of (7) is eliminated. Therefore, the generalized Lyapunov equation is transformed into the following Lyapunov equation:

$$A\hat{P}_1 + \hat{P}_1 A^T + BB^T = 0 \tag{9}$$

An initial solution for the generalized Lyapunov equation is obtained by solving the Lyapunov equation of (9).

Then, in each iteration, the truncated type I controllability Gramians is derived by applying the following iterative formula:

$$A\hat{P}_i + \hat{P}_i A^T + N\hat{P}_{i-1}N^T + BB^T = 0, \quad i = 2,3,\cdots \tag{10}$$

Finally, the type I controllability Gramian is determined as follows:

$$P = \lim_{i\to\infty} \hat{P}_i \tag{11}$$

Similar to the controllability Gramian, the truncated type I observability Gramian can be computed.

Type II Gramians

The generalized Lyapunov equations of the bilinear system can be extended to the following inequality equations [41], which are called type II Gramians:

$$A^T P^{-1} + P^{-1}A + N^T P^{-1} N \le -P^{-1}BB^T P^{-1} \tag{12}$$

$$A^T Q + QA + N^T QN \le -C^T C \tag{13}$$

Although (12) is constructed based on inverse controllability Gramians, it can be rewritten in terms of controllability Gramians by multiplying *P* to left and right-sides of (12).

The type II Gramians have some advantages respect to type I Gramians. These advantages include additional information about the control, global energy bounds, and availability of an H_∞-error bound for the bilinear BT method [34]. Therefore, BT model order reduction for bilinear systems based on type II Gramians are superior to those based on type I Gramians. However, the main difficulty in employing type II Gramians is that finding the type II Gramians via solving the inequalities of (12) and (13), especially for large scale systems is not a straight forward task.

BBT Algorithm

Table 1: Algorithm of bilinear BT method

Input*: The system matrices:* A, N, B, C.

1 Determine low-rank approximation of Gramians:

$$P \approx RR^T \text{ and } Q \approx SS^T;$$

2 Compute SVD of $S^T R$ *as follows:*

$$S^T R = U\Sigma V = [U_1 \quad U_2]\begin{bmatrix}\Sigma_1 & 0\\ 0 & \Sigma_2\end{bmatrix}[V_1 \quad V_2]^T$$

The Σ_1 *contains the r largest singular values of* $S^T R$

3 Construct the transformation matrices T_1 *and* T_2*:*

$$T_1 = SU_1\Sigma_1^{-\frac{1}{2}}$$

$$T_2 = RV_1\Sigma_1^{-\frac{1}{2}}$$

4 Determine the reduced-order bilinear model:

$$A_r = T_2^T A T_1, N_r = T_2^T N T_1, B_r = T_2^T B, C_r = CT$$

Output*:* A_r*,* N_r*,* B_r*,* C_r*.*

Once the type I or type II Gramians are obtained for a bilinear system, they can be employed in the context of balancing for MOR of system. For this purpose, the bilinear BT algorithm can be used as presented in Table 1 [34].

Proposed MOR Method

Numerically, it is not easy to apply balanced truncation to a bilinear system because it requires a solution of two high-order generalized Lyapunov equations. On the other hand, determining the type II Gramians by solving the generalized Lyapunov inequalities is challengeable. Hence, in this paper a new method has been proposed to solve the generalized Lyapunov equations using the LMI approach. After solving the generalized Lyapunov equations by the proposed method, the bilinear BT method is applied to order reduction of bilinear systems. The proposed algorithm is implemented via four steps, as follows.

Step 1. Determining the order of the reduced system:

The first step of determining the reduced-bilinear model of (1) is specifying the desired order. It is necessary to determine which modes have the most significant amount of energy to accomplish this. Modes with higher energy can be evaluated using dominant poles or the Hankel singular value method.

The number of high-energy modes is equal to the order of the reduced model. The order of the reduced model is determined initially by the number of eigenvalues whose real parts are closest to the origin. It is recommended to choose a conservative order at the beginning. The initial order is decreased one by one using a bilinear MOR method such as BPOD [42] until the error index significantly increases.

Therefore, the lowest order with negligible error is the most appropriate.

Step 2. Finding the truncated type II Gramians via LMI:

As stated earlier, in implementing BBT based on type II Gramians, solving the generalized Lyapunov inequalities is complicated, especially for large-scale systems. To address this problem, in this paper, at first, the generalized Lyapunov inequalities would be represented as a LMI constrained optimal problem. Then, it will be solved via an intermediate approximation of the system.

Let us consider the generalized Lyapunov equation as represented by (12). By adding the unknown coefficient of λ to (12), the controllability Gramians equation is converted to the following inequality equation:

$$AP + PA^T + NPN^T + BB^T + \lambda I < 0 \tag{14}$$

On the other hand, the Gramians of controllability should be positive definite. Hence, the controllability Gramians equation can be converted to a constrained optimization problem as follows:

$$\begin{cases} Min\ \lambda \\ s.t.\ AP + PA^T + NPN^T + BB^T + \lambda I_n < 0 \\ P > 0 \end{cases} \tag{15a}$$

By solving the constrained optimization problem of (15), the controllability Gramians would be determined. Similar to controllability Gramians, observability Gramians can be determined, as:

$$\begin{cases} Min\ \mu \\ s.t.\ AQ + QA^T + NQN^T + C^TC + \mu I_n < 0 \\ Q > 0 \end{cases} \tag{15b}$$

In the next step, in order to facilitate solving the optimization problem of (15) and to decrease the computational volume, a middle-order approximation of the original system would be obtained by the conventional MOR methods such as BT or BPOD. It should be noted that both of these methods yields quite precise approximations in middle orders [43]. Then the LMI problem of (15) would be solved for this reduced system. In this case, the optimization problem can be considered as follows:

$$\begin{cases} Min\ \lambda \\ s.t.\ A_mP_m + P_mA_m^T + N_mP_mN_m^T + B_mB_m^T + \lambda I_m < 0 \\ P_m > 0 \end{cases} \tag{16a}$$

$$\begin{cases} Min\ \mu \\ s.t.\ A_mQ_m + Q_mA_m^T + N_mQ_mN_m^T + C_mC_m^T + \mu I_m < 0 \\ Q_m > 0 \end{cases} \tag{16b}$$

where, in (16) index m implies the system matrices and Gramians of the approximated middle-order system.

Step 3. Apply the bilinear BT method:

The achieved type II Gramians in the previous step are applied to the bilinear BT algorithm to obtain a reduced-order bilinear model.

Step 4. Adjust the gain of the reduced-order:

The obtained bilinear reduced-order model by the bilinear BT method usually suffers from steady-state error [44]. In other words, the final value of the achieved bilinear reduced-order model deviates from the final value of the original bilinear system. To address this problem, a feedforward tuning factor is added to the bilinear reduced-order model, which is determined in this step as follows.

Consider the reduced model of (17), in which the tuning factor of K has been added to the output equation as:

$$\begin{cases} \dot{x}_r(t) = A_rx_r(t) + N_rx_r(t)u(t) + B_ru(t) \\ y_r(t) = K\bar{C}_rx_r(t) \end{cases} \tag{17}$$

where $\bar{C}_r$ is the output vector determined by the bilinear BT method in previous step.

The steady-state error of the bilinear BT model is removed by properly adjusting K. To tune this parameter, two approaches can be considered.

In the first approach, the gain of the reduced-order model is tuned as an optimization problem by a swarm intelligence-based algorithm. The fitness function can be considered as a function of output error, such as the Integral Square of Error (ISE).

In the second approach, the tuning factor of K is set so that the steady-state output error for non-oscillating bounded inputs is removed. Let us define the steady-state output error as:

$$\lim_{t\to t_f} e(t) = \lim_{t\to t_f} |y(t) - Ky_r(t)| \tag{18}$$

where, t_f represents the large enough settling time of the response. Then, K is tuned to remove this steady state error as:

$$K = \frac{y(t_f)}{y_r(t_f)} \tag{19}$$

Therefore, the proposed approach can be implemented via the algorithm of Table 2 as follows:

Table 2: The proposed MOR algorithm

Input: *The system matrices:* A, N, B, C.
1 Determine the suitable order of the reduced-bilinear model by an iterative method.
2 Find the middle order approximation of the system and solve the LMI constrained optimization problem of (16) to achieve type II controllability and observability Gramians.
3 Apply the obtained type II controllability and observability Gramians to the BT method to determine the reduced-order bilinear model.
4 Adjust the gain of the system by tuning the factor of K.
Output: A_r, N_r, B_r, C_r.

Simulation Results

Here, three high-order bilinear systems are considered as test systems. These test systems are approximated by the proposed method. A bilinear system with an order of 200 is the first test system. Then, the Chaffee-Infante model would be approximated by the proposed method. The third test system is a nonlinear transmission line circuit converted to a bilinear system by Carleman bilinearization. To validate the proposed method, the obtained reduced-order models are compared with some well-known MOR methods such as BT, BPOD and BIRKA methods. The results show that the proposed method matches the original systems more than other approaches.

Test system 1

In [24], a bilinear system of order 200 is presented with the following matrices:

$$\dot{x}(t) = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ N_1 & 0 \end{bmatrix} x(t)u(t) + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t) \quad (20)$$

$$y(t) = [0 \quad C_2] x(t)$$

where $A_1 \in R^{100\times100}$, $A_2 \in R^{100\times100}$, $B_1 \in R^{100\times1}$ and $C_2 \in R^{1\times100}$

$$A_1 = \begin{bmatrix} -10 & 2 & & \\ 7 & -10 & 2 & \\ & \ddots & \ddots & \ddots \\ & & 7 & -10 \end{bmatrix}, A_2 = \begin{bmatrix} -5 & 2 & & \\ 2 & -5 & 2 & \\ & \ddots & \ddots & \ddots \\ & & 2 & -5 \end{bmatrix} \quad (21)$$

$$N_1 = \begin{bmatrix} 2 & 1 & & \\ -1 & 2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & -1 & 2 \end{bmatrix}, B_1 = [1 \quad \cdots \quad 1],\ C_2 = [0 \quad \cdots \quad 0 \quad 1]^T$$

The procedure of order reduction by the proposed method is followed step by step as:

Step 1: In the first step, the order of the reduced bilinear model is determined. For this purpose, the initial order is determined based on the number of eigenvalues with the real part close to the origin. According to Fig. 1, this guess is 20. Then, the order of the reduced-model decreased one by one from 20 to 1, and for each order, the BPOD was applied. The H_2 norm of error for each order is calculated. The H_2 norm of the error for each order of the reduced test system 1 is shown in Fig. 2.

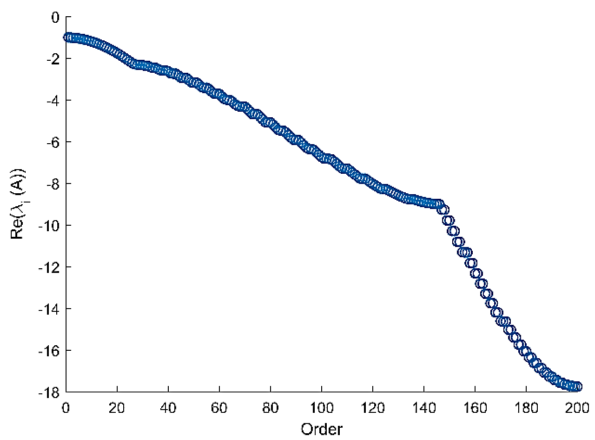

Fig. 1: Real part of eigenvalues of test system 1.

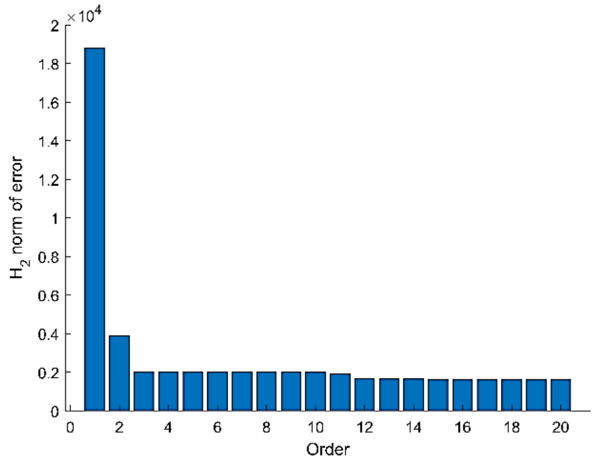

Fig. 2: H_2-norm of error versus order.

It can be shown that the proper order for the reduced-order bilinear model of test system 1 is 2.

Step 2: In this step, a new structure for computation of controllability and observability Gramians would be considered. By adding an unknown term of α to the controllability relation, this equation is converted to a LMI optimization problem. A similar approach can be applied to observability Gramians. Then, the optimization problem with inequality constraints of (15) is minimized to determine the alternative controllability Gramian. Following the same procedure, the observability Gramian is determined, as well.

Step 3: Using the bilinear BT method, the reduced-order bilinear model is derived.

Step 4: Adjust the gain of the reduced-order system to remove the steady-state error. Here $K = 1.016$ has selected as the ratio of the original and reduced order systems.

The achieved reduced-order bilinear model is presented as follows:

$$\dot{x}_r(t) = \begin{bmatrix} -1.0196 & -7.0814e-18 \\ -1.3391e-18 & -1.0092 \end{bmatrix} x_r(t) + \begin{bmatrix} 0 & 0 \\ 3.2332 & -1.1425e-18 \end{bmatrix} x_r(t)u(t) + \begin{bmatrix} 4.9068 \\ 2.5369e-17 \end{bmatrix} u(t) \quad (22)$$

$$y_r(t) = 1.016 \times [0 \quad 12.1142] x_r$$

Fig. 3 illustrates the response of the reduced bilinear model of Eq. (22) to input $u(t) = 0.05exp(-0.5t)$. Also, it is compared with some well-known MOR methods, including BT, BPOD and BIRKA methods. In addition, the absolute error has been also evaluated over time and depicted in Fig. 4. It can be shown that the proposed method matches the original system much better than other methods, and it has a smaller error compared to the other methods. Some important characteristics of the response are compared to provide a quantitative and numerical evaluation. These specifications include peak, steady-state, and ISE index as an appropriate measures for evaluating the approximation error. The results of the comparison have been presented in Table 2.

Figs. 3, 4 and Table 3 show that the obtained reduced-order bilinear model via the proposed method provides the best approximation among the other indicated approaches.

Test System 2: Chaffee Infante

Another standard test system used to evaluate MOR methods is the one-dimensional Chaffee-Infante equation on $\Omega = (0, L) \times (0, T)$ [42]. In this equation, there is cubic nonlinearity:

$$v_t + v^3 = v_{xx} + v \quad (0, L) \times (0, T) \tag{23}$$

where v is the viscosity parameter.

The initial and boundary conditions of the Chaffee-Infante equation have been considered in (24)-(26).

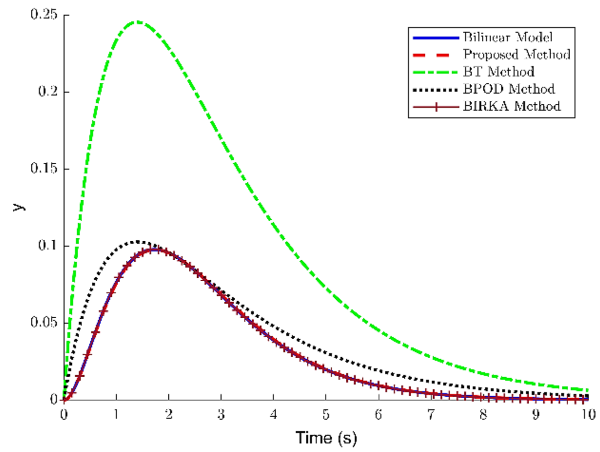

Fig. 3: Comparison of responses of the bilinear model of test system 1 and their reduced-order model approximations.

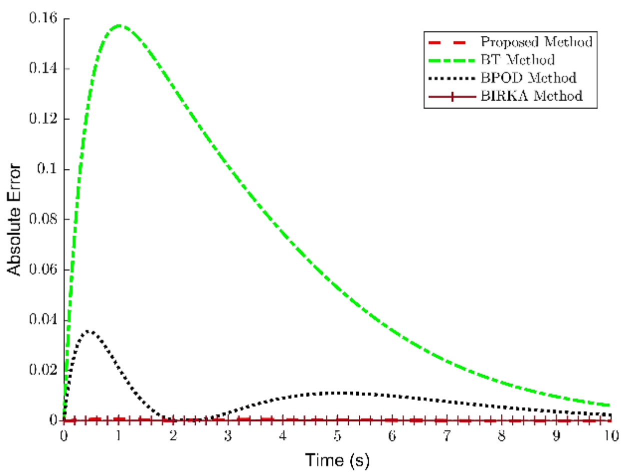

Fig. 4: Time evolution of absolute error of various methods for approximations of test system 1.

Table 3: Comparison of methods for test system 1

	Order	Final value	Peak	ISE
Original System	200	3.24e-04	0.0977	-
Proposed Method	2	3.31e-04	0.0975	1.02e-06
BT Method	2	0.0063	0.2453	0.0660
BPOD Method	2	0.0027	0.1027	0.0013
BIRKA Method	2	3.19e-04	0.0978	1.63e-08

$$\alpha v(0, t) + \beta v(0, t) = u(t) \tag{24}$$

$$v(L, t) = 0 \quad t \in (0, T) \tag{25}$$

$$v(x, 0) = v_0(x) \tag{26}$$

where α and β are constant parameters and $v_0(x)$ is initial condition of the system.

A finite-difference scheme was used to obtain the spatial discretization system. Then, Carleman bilinearization converts the nonlinear ODEs of the Chaffee-Infante equation to bilinear form. For this test system, $L = 0.1$ and $T = 5$. Also, the initial condition is considered zero, i.e., $v_0(x) = 0$. The discretization involved 31 points. Thus, the order of the bilinear model of the Chaffee-Infante is 992. The proposed method is applied to order reduction of the bilinear Chaffee-Infante model. The order of reduced approximation is 10. Similar to test system 1, the obtained reduced-order model is compared with some well-known model order reduction methods such as BPOD, BBT and BIRKA methods. In Fig. 5, responses of reduced-order bilinear models to input $u(t) = 0.5(1 + \cos(\pi t))$ are shown. Also, the absolute error versus time is presented in Fig. 6.

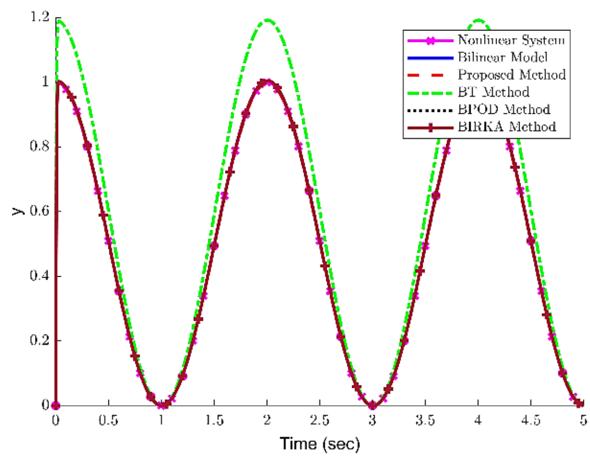

Fig. 5: Comparison of responses of the bilinear model of Chaffee Infante equation and their reduced-order model approximations.

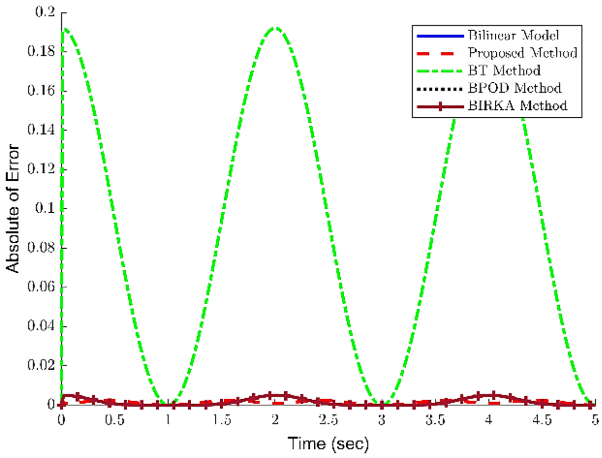

Fig. 6: Time evolution of absolute error of various methods for bilinear Chaffee Infante approximations.

According to Fig. 5 and Fig. 6, it is seen that the proposed method results as well as those of BIRKA are similar to the high-order bilinear model of the Chaffee Infante model.

Test system 3: Transmission Line Circuit

Fig. 7 depicts the transmission line circuit, including nonlinear resistors and an independent current source.

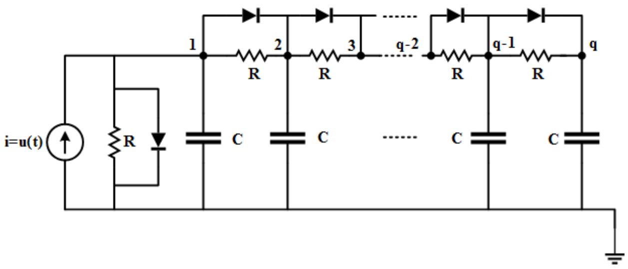

Fig. 7. Transmission line circuit.

Transmission lines can be modeled as a nonlinear state spaces form, as follows [6]:

$$\dot{x}(t)=\begin{bmatrix}-g(x_1)-g(x_1-x_2)\\-g(x_1-x_2)-g(x_2-x_3)\\\vdots\\-g(x_{k-1}-x_k)-g(x_k-x_{k+1})\\\vdots\\-g(x_{q-1}-x_q)\end{bmatrix}+\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}u(t) \quad (27)$$

$$y(t)=[1 \quad 0 \quad \cdots \quad 0]x(t)$$

where $x\in R^{q\times 1}$ is the state variables, $f\in R^q$ is nonlinear state evolution function, $b\in R^{q\times 1}$ and $c\in R^{1\times q}$ are input and output, respectively. Also, the relation between voltage and current of each resistor is modeled as $g(x)=exp(x)+x-1$.

In this case, the number of resistors is chosen to be 20. In order to approximate the nonlinear RC circuit system (27) using a bilinear model, the Carleman bilinearization is used [4], [5]. The order of the resulting bilinear model is $q^2+q=420$.

The obtained bilinear model of the transmission line is high-order and should be reduced. For this purpose, the proposed method is applied to approximate the reduced-order bilinear model of the transmission line model. The reduced-order bilinear model for the bilinear transmission line model is as follows:

$$\dot{x}_r(t)=\begin{bmatrix}-68.20 & -9.87 & -3.11\\-6.42 & -92.79 & 9.69\\2.96 & 18.77 & -61.27\end{bmatrix}x_r+\begin{bmatrix}0.011 & 0.038 & -0.039\\0.038 & 0.181 & -0.133\\-0.008 & -0.03 & 0.024\end{bmatrix}x_r u+\begin{bmatrix}-0.09\\-0.26\\0.18\end{bmatrix}u \quad (28)$$

$$y_r(t)=20\times[-0.06 \quad -0.18 \quad 0.112]x_r$$

The responses of the nonlinear transmission line model and their approximations to input $u(t)=sin(10t)cos(t)exp(-1.5t)$ have been presented in Fig. 8. Also, the absolute error of the obtained reduced-order models has been shown in Fig. 9. In Table 4, some specifications of the achieved reduced-order bilinear model have been compared, as well. It is seen from Fig. 8 and Fig. 9 that the proposed method and the BIRKA have the most similarity and less error among the reduced-order bilinear systems. Besides, according to Table 4, it can be observed that the proposed method and the BIRKA method have the best approximation among the reduction methods.

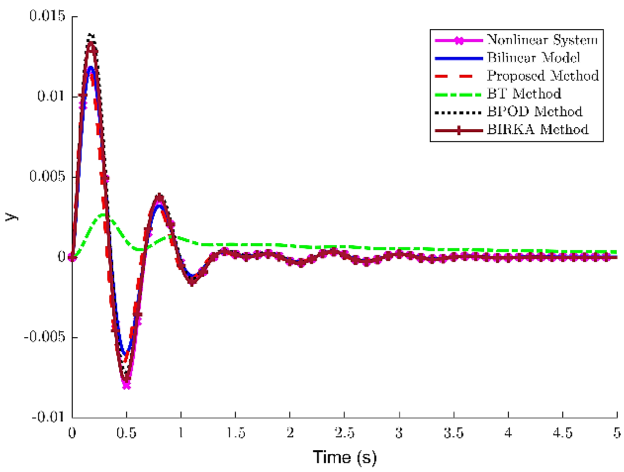

Fig. 8: Comparison of responses of the nonlinear transmission line system and their reduced-order model approximations.

However, although the BIRKA is as accurate as the proposed method, its convergence is not guaranteed, generally [39].

In 50 simulations performed by the BIRKA to test System 3, it was observed that BIRKA diverges nine times, indicating an 18% failure rate in model order reduction. It is the main drawback of BIRKA, which is not observed in the BT family.

Indeed, as discussed earlier, convergence is guaranteed via employing BT. Therefore, the proposed method improves the bilinear BT method's performance and preserves the bilinear BT method's specifications, such as guaranteed stability and convergence.

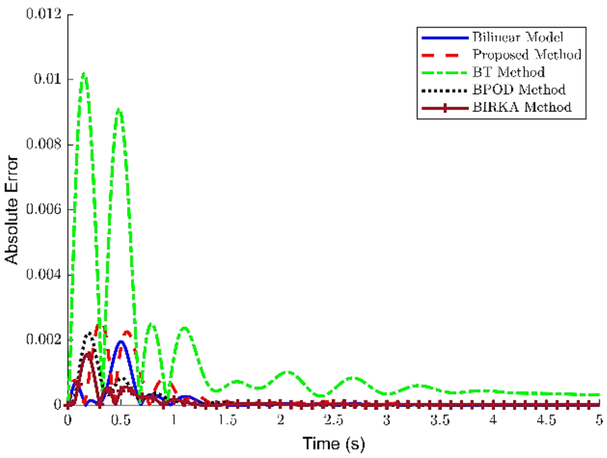

Fig. 9: Time evolution of absolute error of various methods for bilinear transmission line model approximations.

Table 4: Comparison of methods for test system 3

	Order	Final value	Peak	ISE
Nonlinear pendulum system	20	1.155e-05	0.0118	-
Bilinear Model	420	1.330e-05	0.0118	6.43e-07
Proposed Method	3	8.247e-07	0.0118	1.83e-06
BT Method	3	3.283e-04	0.0026	3.30e-05
BPOD Method	3	7.426e-07	0.0140	8.04e-07
BIRKA Method	3	-1.2903e-06	0.0134	3.51e-07

To further analysis, the simulation time of the MOR methods is compared. The time required for approximation of test system 3 by MOR methods is given in Table 5.

Table 5: Comparison of the simulation time of MOR methods for test system 3

	Simulation Time (sec)	Quality of Approximation
Proposed Method	6.24	Very good
BT Method	6.81	Not good
BPOD Method	32.69	Good
BIRKA Method	431.33	Very good

It can be seen that the proposed method and the bilinear BT method need less time to approximate the test system 3. However, the bilinear reduced-order model obtained by the bilinear BT method is not a good approximation. On the other hand, the proposed method and BIRKA have high quality to approximate test system 3, but the BIRKA needs about 70 times more simulation time.

It can be noted that the proposed method used to MOR of test system 3 is implemented by minimizing the optimization problem of (16). To do this, an initial bilinear reduced-order model with order 25 is approximated by a bilinear BT method. Then, the optimization problem of (16) is minimized to determine the bilinear reduced model. The required simulation time for this two-stage is 6.24 seconds.

Results and Discussion

This study investigates the MOR of the bilinear systems based on the improved bilinear BT method. The proposed method uses the concept of type II Gramians to determine controllability and observability Gramians. To determine these Gramians, a new LMI-based approach is applied. After determining the new Gramians, the bilinear BT method is used. Since type II Gramians have more advantages than type I Gramians, the accuracy of the obtained reduced-order bilinear model is higher than the bilinear BT method. The proposed method is not only more accurate than bilinear BT, but it also has the advantages of balanced truncation, including ensuring stability and convergence.

Furthermore, the steady-state error of the reduced-order bilinear model is removed by adjusting the tuning factor. Three test systems are considered and compared with some well-known MOR methods to evaluate the proposed method. The results show that the proposed method is more similar to high-order bilinear systems and outperforms other approaches.

Conclusions

This paper proposes a new MOR method based on the balance truncation approach with type II Gramians for order reduction of the bilinear systems. For this purpose, at first, a new iterative method is proposed for determining the proper order for the reduced-order bilinear model which is related to the number of eigenvalues of the bilinear system whose real parts are closest to origin and have the most significant amount of energy. Then, the generalized Lyapunov equations are constructed to determine the Gramians of controllability and Gramians of the observability. These generalized Lyapunov equations are transformed into a linear matrix inequality problem by adding an unknown coefficient. Next, the LMI problem is converted to a constrained optimization problem. New controllability and observability Gramians are determined by solving the constrained LMI optimization problem. The obtained Gramians are applied to the bilinear BT method to determine the reduced-order bilinear model.

These type II Gramians, determined by solving the constrained optimization problem as an LMI problem, contain additional information compared to type I Gramians. Also, type II Gramians lead to finding global energy bounds. Therefore, obtaining type II Gramians in the context of balancing leads to increasing the accuracy of the BT method. On the other hand, the order determined by the proposed method is more appropriate and accurate than other methods of determining the order of reduced systems. Three high-order bilinear test systems are approximated to show the efficiency and ability of the proposed method. The achieved results are compared with some classical order reduction methods such as the BT, BPOD and BIRKA. According to the obtained results, it can be concluded that the proposed MOR method is superior to classical bilinear MOR methods, but is almost equivalent to BIRKA. It is out-performance respecting BIRKA is its guaranteed stability and convergence.

Author Contributions

H. Nasiri Soloklo designed and simulated the proposed method and wrote the manuscript. N. Bigdeli chose strategies, analyzed the results, edited the manuscript, and managed the entire process.

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Conflict of Interest

The authors declare no potential conflict of interest regarding the publication of this work. In addition, the ethical issues including plagiarism, informed consent, misconduct, data fabrication and, or falsification, double publication and, or submission, and redundancy have been completely witnessed by the authors.

Abbreviations

BBT	Bilinear Balanced Truncation
BIRKA	Bilinear Iterative Rational Krylov Subspace
BPOD	Bilinear Proper Orthogonal Decomposition
ISE	Integral square of Error
LMI	Linear Matrix Inequality
MOR	Model Order Reduction

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