Determination of the Maximum Dynamic Range of Sinusoidal Frequencies in A Wireless Sensor Network with Low Sampling Rate

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Abstract
Background and Objectives: Subsampling methods allow sampling signals at rates much lower than Nyquist rate by using low-cost and low-power analog-to-digital converters (ADC). These methods are important for systems such as sensor networks that the cost and power consumption of sensors are the core issue in them. The Chinese remainder theorem (CRT) reconstructs a large integer (input frequency) from its multiple remainders (aliased or under-sampled frequencies), which are produced from under-sampling or integer division by several smaller positive integers. Sampling frequencies can be reduced by approaches based on CRT.

Methods: The largest dynamic range of a generalized Chinese remainder theorem for two integers (input frequencies) has already been introduced in previous works. This is equivalent to determine the largest possible range of the frequencies for a sinusoidal waveform with two frequencies which the frequencies of the signal can be reconstructed uniquely by very low sampling frequencies. In this study, the largest dynamic range of CRT for any number of integers (any number of frequencies in a sinusoidal waveform) is proposed. It is also shown that the previous largest dynamic range for two frequencies in a waveform is a special case of our proposed procedure.

Results: A procedure for multiple frequencies detection from remainders (under-sampled frequencies) is proposed and maximum tolerable noises of undersampled frequencies for unique detection is obtained. The numerical examples show that the proposed approach, in some cases, can gain 11.5 times higher dynamic range than the conventional methods for a multi-sensor under-sampling system.

Conclusion: Other studies introduced the largest dynamic range for the unique reconstruction of two frequencies by CRT. In this study, the largest dynamic ranges for any number of frequencies are investigated. Moreover, tolerable noise is also considered.
in [5] to estimate frequencies when a signal is under sampled by multiple under-sampling frequencies.

In [6], the statistical model of CRT-based multiple parameter estimation is investigated, and two approaches are introduced to address the problems of ambiguity resolution in parameter estimation.

A method based on CRT is introduced in [7] to estimate the direction of arrival (DOA) of the signal. This algorithm has less complexity with similar precision in comparison with other algorithms for DOA estimation.

In [8], CRT and non-orthogonal multiple access (NOMA) techniques are introduced for unmanned aerial vehicle (UAV) relay networks to improve communication between transmitter and receiver.

A combination of CRT with Haar Wavelet Transform was proposed as a watermarking technique in [9] to hide information.

The Haar Wavelet Transform has been used for imperceptibility, and CRT provides the security of the watermarked image. A reversible sketch data structure based on CRT was proposed in [10] to compress and fuse big data network traffic. In [11], a novel CRT-based conditional privacy was introduced to keep an authentication scheme for securing vehicular authentication.

In this work, the CRT can help the trusted authorities to generate and broadcast group keys to the network vehicles.

In [12], the authors proposed a multiple secret image-sharing scheme by CRT and Boolean exclusive-OR operation.

A robust and secure data-hiding method in the Tchebichef domain is presented based on CRT [13]. The efficiency of the algorithm was confirmed by implementing the algorithm over different images.

Power efficiency is one of the critical design factors in wireless sensor network systems. In such systems, it is possible to digitalize received analog signal by sensors with very low frequencies and use CRT for manipulation and reconstruction of the frequencies of the main signal. In [14] a low-frequency power efficient digital signal processing architecture for mathematic operations based on CRT was designed and implemented.

A packet forwarding scheme based on CRT was developed for wireless sensor networks in [15]. The advantages of this scheme are energy efficiency, low computational complexity and high reliability.

In [16], an approach based on the frequency domain sparse common support and CRT was developed for frequency determination of multiple sinusoidal signals when the sampling rate even less than Nyquist rate. Authors in [17] proposed an approach to reconstruct the multiple frequencies of a sinusoidal waveform from aliased frequencies by the CRT approach.

In all these researches the dynamic range for unambiguously reconstruction integers (e.g. frequencies), which are divided by a set of modules (e.g. sampling frequencies) from their remainders (e.g. aliased frequencies) is important.

The higher dynamic range for a set of modules means the possibility to reconstruct the larger range of integers un-ambiguously by remainder of integers from those modules. Thus, any improvement in the dynamic range will lead to more efficient schemes in many applications [3].

The dynamic range for the unique determination of an integer (frequency) \( N \), with modules (sampling frequencies) \( \Gamma = \{m_1, m_2, \ldots, m_r\} \) is the least common multiple (lcm) of modules i.e., \( d = lcm(m_1, m_2, \ldots, m_r) \) [5], [18]. A dynamic range for the unique determination of two integers (frequencies) \( N_1 \) and \( N_2 \) can be obtained as \( d = \min\{\max\{I_1, I_2\}\} \) where \( I_1 \cup I_2 = \Gamma \) [19]. The first generic dynamic range for reconstruction of multiple integers (more than two integers (frequencies)) from their modules was introduced in [20].

A sharpened dynamic range for \( \rho \) integers (\( \rho = 1,2,\ldots \)) was presented as \( d = \min\{\max\{I_1, \ldots, I_\rho\}\} \) where \( I_1 \cup \ldots \cup I_\rho = \Gamma \) in [19]. Dynamic ranges for multiple integers when there are conditions over integers are presented in [21], [22].

The largest dynamic range for two integers is obtained as \( d = \min\{I_1 + I_2\} \) where \( I_1 \cup I_2 = \Gamma \) in [23] and the maximum tolerable error for two integers was discussed in [24] and it is applied in [25] for the direction of arrival (DOA) of two sources and in [26], [27] was used for secret image sharing by the modular operation.

Most of the previous studies discussed the unambiguous dynamic range for two integers (frequencies) or assumed conditions for integers (frequencies) [21], [22], [28] that will be discussed with details in the background section while we present a close form relationship of the largest dynamic range for multiple integers (frequencies) without condition on them. Furthermore, we show that the largest dynamic range for reconstruction of two integers (frequencies) is a special case of our work.

The presentation is organized as follows. Related works with theoretical background is discussed in Background section.

A proposition for finding the maximum possible range for unique reconstruction of any number of input frequencies from under-sampled frequencies is introduced in Proposed Approach section. Furthermore, the proposed proposition is specified for two and three input frequencies in corollaries, the maximum tolerable noise for the maximum possible range is obtained and a
procedure for reconstruction is also introduced in the Proposed Approach section.

Different numeric examples to verify the effectiveness of the proposed approaches are introduced in the Simulation Results section.

Finally, the work is concluded in the Conclusion section.

Background

Consider a complex waveform without noise as follows [23]:

\[ x(t) = \sum_{i=1}^{\rho} A_i e^{i(2\pi f_i t + \varphi_i)} + w(t) \]  

(1)

where \(A_i\)'s are unknown nonzero complex coefficients and \(f_i\)'s; \(1 \leq l \leq \rho\) are multiple unknown frequencies in Hz that should be determined. The \(w(t)\) is additive white Gaussian noise.

Consider \(\gamma\) sensors in a wireless sensor network with \(\Gamma = \{f_{s1}, f_{s2}, \ldots, f_{s\gamma}\}; \gamma \geq 2\) sampling rates as Fig. 1 in which all may be much less than the unknown frequencies i.e. \(f_{si} < f_i; i = 1, \ldots, \gamma; l = 1, \ldots, \rho[1, 2]\).

Assume these sampling frequencies are co-prime i.e. \(M = \text{lcm}(f_{s1}, f_{s2}, \ldots, f_{s\gamma}) = f_{s1}f_{s2} \ldots f_{s\gamma}\) and without loss of generality assume \(f_{s1} < f_{s2} < \ldots < f_{s\gamma}\) that \(\text{lcm}\) is the least common multiplier.

Then, the multiple under-sampled waveforms by sampling frequency \(f_{sr}; r = 1, \ldots, \gamma\) are given by [23]:

\[ x_{fsr}(n) = \sum_{i=1}^{\rho} A_i e^{2\pi in f_{si}} \]  

(2)

Using the \(f_{sr}\)-point discrete Fourier transform (DFT) to \(x_{fsr}(n)\), relation (2) can be written as:

\[ \text{DFT}_{fsr}(x_{fsr}(n))[k] = \sum_{l=1}^{\rho} A_l \delta(k - f_{u(l,r)}) \]  

(3)

\[ 1 \leq r \leq \gamma \]

where \(\delta(k)\) is equal to 1 when \(k = 0\) and others \(\delta(k) = 0\). The \(f_{u(l,r)}\) is remainder (under-sampled frequency) of \(F_i\) with module (sampling frequency) \(f_{sr}\) i.e. \(f_{u(l,r)} = F_i \mod f_{sr}\).

Thus, following under-sampled frequencies set \(S_r(F_1, \ldots, F_\rho)\) can be written.

\[ S_r(F_1, \ldots, F_\rho) = \bigcup_{l=1}^{\rho} \{f_{u(l,r)}\}, \]  

\[ r = 1, \ldots, \gamma \]

Consider \(F_{\max}\) be an upper bound of input frequencies when all input frequencies less than \(F_{\max}\) (i.e., \(F_i \leq F_{\max}\), \(l = 1, \ldots, \rho\) can be uniquely reconstructed from their remainders. Some works have been done to determine \(F_{\max}\) for unambiguous reconstruction of multiple integers (multiple frequencies) from their remainders sets where we briefly review them in the sequel.

**Proposition 1** [29, 30]: A large dynamic range \((F_{\max})\) for unique determination \(F_i, l = 1, \ldots, \rho\) when under-sampled with \(f_{sr}, r = 1, \ldots, \gamma\) is \(F_{\max} = \max(u, f_{sr})\) where \(u = \min_{1 \leq sl_1 \leq \ldots \leq sl_\gamma} \text{lcm}(f_{sr(sl_1)}, \ldots, f_{sr(sl_\gamma)})\) that \(\gamma = \eta \rho + \theta\) for some \(0 \leq \theta < \rho\).

A proposed majority method for the determination of multiple integers from their moduli introduced in [19] as follows:

**Proposition 2** [19]: A large dynamic range for multiple integers (multiple frequencies) is \(F_{\max} = \min_{i_1 \cup \ldots \cup i_p = \Gamma} \max(\prod_{i \in i_1} f_{si}, \ldots, \prod_{i \in i_p} f_{si})\) where \(i_1, \ldots, i_p\) are disjoint set where \(l_1 \cup \ldots \cup l_p = \Gamma\) and \(l_1 \cap l_j = \varphi\) for \(1 \leq i \neq j \leq \rho\) and \(l_i\) can be an empty set.

**Proposition 3** [23]: For two integers \(\rho = 2\) as \((F_{u1}, F_{u2})\) with moduli \(\Gamma = (f_{s1}, \ldots, f_{s\gamma})\) the largest dynamic range for unambiguous reconstruction from remainders is \(F_{\max} = \min_{i_1, i_2} \{\text{lcm}(i_1), \text{lcm}(i_2)\} = \min_{i_1, i_2} (\prod_{i \in i_1} f_{si} + \prod_{i \in i_2} f_{si})\) where \(i_1 \cup i_2 = \Gamma\).

The largest dynamic range for single integer \((\rho = 1)\) is \(\text{lcm}\) of all moduli i.e. \(F_{\max} = \text{lcm}(f_{s1}, \ldots, f_{s\gamma})\). It can be inferred from Proposition 1-3 that the dynamic range for multiple integers, in general, is less than \(\text{lcm}\) of all modules.

Thus, some works [22, 28] tried to achieve maximal possible range similar to single integer i.e \(\text{lcm}\) of all

moduli.

To do this, they used some conditions on the multiple integers (input frequencies) or /and moduli (under-sampling frequencies) that was reviewed briefly at the following.

**Proposition 4:** If \( F_0 - F_1 < f_{s1}/2 \) then \( F_{max} = \text{lcm}(f_{s1}, \ldots, f_{sy}) = \prod_{i=1}^{y} f_{si} \).

Note that this paper achieves \( \text{lcm} \) of all moduli while the condition \( F_0 - F_1 < f_{s1}/2 \) is admitted.

**Proposition 5 [22]:** If \( F_0 - F_1 < f_{s2} \), GCD(\( \rho_s, f_{s2} \)) = 1; \( 1 \leq i \leq \gamma - 1 \) and \( \rho^2 - \rho(f_{s1} + l) + (l - 1)f_{sy} > 0 \) for \( 2 \leq l \leq \rho \) then \( F_{max} = \text{lcm}(f_{s1}, \ldots, f_{sy}) = \prod_{i=1}^{y} f_{si} \) where \( f_{s1} < \ldots < f_{sy} \) and GCD is the greatest common division.

Note that in this case, the difference between two disjoint integers (input frequencies) should be less than the minimum sampling frequency \( f_{s1} \) (i.e. \( F_0 - F_1 < f_{s1} \)) while for Proposition 4, it is \( f_{s1}/2 \).

A multiple frequencies determination for narrow bandwidth signals when the maximum difference between input frequencies (multiple integers) are less than the maximum sampling frequency (moduli) i.e. \( F_0 - F_1 < f_{sy} \) was also proposed in [21].

**Proposed Approaches**

**Lemma 1 [23]:** If two input frequencies sets \( X = \{F_1, \ldots, F_p\} \) and \( Y = \{F_1', \ldots, F_p'\} \) have the same remainder sets, i.e. \( S_r(X) = S_r(Y) \), then the minimum of these integers would be zero, i.e. \( \min\{X \cup Y\} = 0 \), and the maximum of these integers would be \( \max\{X \cup Y\} = F_{max} \) that \( F_{max} \) is a large dynamic range.

The order of remainders of each modulus is not known from output of DFT [19].

In other words if the ordered remainders set of multiple integers for \( r^{th} \) modulus be \( S_r(F_1, \ldots, F_p) \), then received the remainders set from output of DFT is \( S_r(F_1, \ldots, F_p) = \bigcup_{r=1}^{\rho} \{f_{u(l,r)}\}, r = 1, \ldots, \gamma \) where \( \delta_r \) is an arbitrarily chosen onto mapping from index set \( T = \{1, \ldots, \rho\} \) to the indices of elements in \( S_r(F_1, \ldots, F_p) \). Note that when \( \delta_r(l) = l \) we have \( t_{u(\delta_r(l),r)} = t_{u(l,r)} = f_{u(l,r)} \).

It means \( l^{th} \) remainder of modulus \( r \) in ordered set of DFT, i.e. \( S_r \), corresponding to the \( i^{th} \) integer, i.e. \( F_i \), (See Table. 1).

<table>
<thead>
<tr>
<th>Integer</th>
<th>Mod ( f_{s1} )</th>
<th>Mod ( f_{s2} )</th>
<th>...</th>
<th>Mod ( f_{sy} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_1' )</td>
<td>( t_{u(\delta_1(1),1)} )</td>
<td>( t_{u(\delta_2(1),2)} )</td>
<td>...</td>
<td>( t_{u(\delta_y(1),y)} )</td>
</tr>
<tr>
<td>( F_2' )</td>
<td>( t_{u(\delta_1(2),1)} )</td>
<td>( t_{u(\delta_2(2),2)} )</td>
<td>...</td>
<td>( t_{u(\delta_y(2),y)} )</td>
</tr>
</tbody>
</table>
| ... | ... | ... | ... | ...
| \( F_{\rho'} \) | \( t_{u(\delta_1(\rho),1)} \) | \( t_{u(\delta_2(\rho),2)} \) | ... | \( t_{u(\delta_y(\rho),y)} \) |

We try to find integers \( F_i' \) based on its remainders \( t_{u(\delta_r(l),r)} = 1, \ldots, \gamma \) (see Table. 1). The relationship between an integer \( F_i \) and its remainders is as follows:

\[
F_{u(l,r)} = F_i \mod f_{sy} = F_i - k_{ir}f_{sr}
\]  
(5)

where \( k_{ir} \in [0, 1, \ldots, |F_{max}/f_{sr}|] \).

The relationship between moduli \( f_{sr} \); \( r = 1, \ldots, \rho \) and \( f_{u(l,r)} \) as the remainder of \( F_i \) is as follows:

\[
f'_{u(l,r)} = F'_i \mod f_{sr} = F'_i - k'_{ir}f_{sr}
\]  
(6)

**Proposition 6:** Assume two frequencies sets \( X = \{F_1, \ldots, F_p\} \) and \( Y = \{F_1', \ldots, F_p'\} \) have the same under-sampled (remainder) sets i.e. \( S_r(X) = S_r(Y) \) with sampling frequencies (moduli) \( f_{sr}, r = 1, \ldots, \gamma \). Now assume from all \( \gamma \) remainders for each \( F_i \in X \) and \( F_i' \in Y \) there are \( \alpha_{(l,i)} \); \( l = 1, \ldots, \rho \); \( i = 1, \ldots, \rho \) common remainders (same remainders) with moduli \( f_{sr(h)} \); \( h = 1, \ldots, \alpha_{(l,i)} \) between \( F_i \in X \) and \( F_i' \in Y \). Then the difference value between \( F_i \in X \) and \( F_i' \in Y \) is as follows:

\[
F'_i - F_i = k_{li}_r \text{lcm}(\bigcup_{h=1}^{\alpha_{(l,i)}} f_{sr(h)})
\]  
(7)

**Proof of Proposition 6:** There are \( \alpha_{(l,i)} \) same remainders between \( F'_i \) and \( F_i \) thus difference between these \( \alpha_{(l,i)} \) remainders should be zero i.e. \( f_{u(lsr(h))} - f_{u(lsr'(h))} = 0 \); \( h = 1, \ldots, \alpha_{(l,i)} \). By considering (5) and (6) we have \( f_{u(lsr(h))} - f_{u(lsr'(h))} = F'_i - F_i - k'_{ir}f_{sr(h)} = 0 \) where \( k'_{ir} \in [0, 1, \ldots, \pm |F_{max}/f_{sr}|] \). So, it is possible to have following relationships:
\( F'_i - F_i = k'' \ell_{F'_h} f_{sr_{r1}(i)} \)
\( = \ldots = k'' \ell_{F'_h} f_{sr_{r1}(i)} = \ldots = k'' \ell_{F'_h} f_{sr_{r1}(i)} = A \) \hfill (8)

From [8] it is obvious that \( A \) should be multiple of \( \alpha_{li} \) moduli frequencies, i.e. \( A/f_{sr_{r1}(i)} = k'' \ell_{F'_h} f_{sr_{r1}(i)} \); \( h = 1, \ldots, a_{li} \). Therefore, the smallest value that is divisible to all moduli frequencies \( f_{sr_{r1}(i)}, \ldots, f_{sr_{r1}(i)}, f_{sr_{r1}(i)} \) is the least common multiple (\( lcm \)) of them i.e. \( lcm(f_{sr_{r1}(i)}, \ldots, f_{sr_{r1}(i)}, f_{sr_{r1}(i)}) \). Thus, \( A \) is multiple of \( lcm \) of \( \alpha_{li} \) moduli frequencies i.e. \( A = k_{li} lcm(f_{sr_{r1}(i)}, \ldots, f_{sr_{r1}(i)}) \). From Lemma 1 it is clear that \( F'_i, F_i \in [0, F_{max}] \) thus \( A = F'_i - F_i = k_{li} lcm(\bigcup f_{sr_{r1}(i)}); k_{li} \in \) \( (0, \pm 1, \ldots, \pm F_{max} \bigcup lcm(f_{sr_{r1}(i)}, \ldots, f_{sr_{r1}(i)}) \)].

**Proposition 7:** Assume a set of \( \rho \) frequencies (integers) as \( \{F'_1, \ldots, F'_\rho\} \) from under-sampled frequencies (remainders) with sampling frequencies (moduli) \( f_{sr_{r1}} \) have the same remainders as follows:

\[
F'_i = F_i = k'' \ell_{F'_h} f_{sr_{r1}(i)} \quad i = 1, \ldots, \rho
\]

where \( Y = \{F'_1, \ldots, F'_\rho\} \) have the same remainders as sets \( X = \{F_1, \ldots, F_\rho\} \) have the same remainder sets with moduli \( f_{sr_{r1}} \). The maximum \( x \) value \( \leq \gamma \) where \( max(x) \leq F_{max} \) and \( F_{max} \) is the number of common remainders between \( F'_i \) and \( F_i \) and \( k_{li} \in (0, \pm 1, \ldots, \pm F_{max} \bigcup lcm(f_{sr_{r1}(i)}, \ldots, f_{sr_{r1}(i)}) \] and \( \rho \) is the set of all moduli \( \Gamma = \bigcup f_{sr_{r1}} \) and based on Proposition 6 the difference value is equal to \( \gamma \). Thus, we can say \( F'_i - F_i = k_{li} lcm(\bigcup f_{sr_{r1}(i)}) \) is the largest dynamic range for \( \{F'_1, \ldots, F'_\rho\} \) and all \( F_i ' \)s are sampled frequencies (moduli) as \( \gamma \) can be reconstructed unambiguously when \( max(x) \leq F_{max} \) where \( F_{max} \) is called the largest dynamic range.

The largest dynamic range for \( \rho \) integers from remainders (frequencies) with moduli (sampling frequencies) \( f_{sr_{r1}} \) can be obtained as follows:

\[
F_{max} = \max(\{F_1, \ldots, F_\rho\})
\]

\[
\sum_{i=1}^{\rho} \sum_{l=1}^{\rho} \left| F'_i - F_i - k_{li} lcm(\bigcup f_{sr_{r1}(i)}) \right| = 0
\]

where \( Y = \{F'_1, \ldots, F'_\rho\} \) have the same remainders as \( X = \{F_1, \ldots, F_\rho\} \) and \( F_{max} \) is the number of common remainders between \( F'_i \) and \( F_i \) and \( k_{li} \in (0, \pm 1, \ldots, \pm F_{max} \bigcup lcm(f_{sr_{r1}(i)}, \ldots, f_{sr_{r1}(i)}) \] and \( \rho \) is the set of all moduli \( \Gamma = \bigcup f_{sr_{r1}} \) and based on Proposition 6 the difference value is equal to \( \gamma \). Thus, we can say \( F'_i - F_i = k_{li} lcm(\bigcup f_{sr_{r1}(i)}) \) is the largest dynamic range for \( \{F'_1, \ldots, F'_\rho\} \) and all \( F_i ' \)s are sampled frequencies (moduli) as \( \gamma \) can be reconstructed unambiguously when \( max(x) \leq F_{max} \) where \( F_{max} \) is called the largest dynamic range.

In the following proposed procedure is introduced to obtain the largest dynamic range \( F_{max} \) from \( \gamma \). Note that when all under-sampling frequencies multiplied by constant \( c \) (increased \( c \) times) the \( lcm \) of under-sampling frequencies are also multiplied by \( c \). Then, the maximum possible frequencies that satisfied \( \gamma \) i.e. \( F_{max} \) will be also multiplied by \( c \).

**Procedure 1:** The procedure for determination of the largest dynamic range can be summarized as follows:

\[ \text{Step 0: Initialize the largest dynamic range as } F_{max} = F_{\text{init}} \text{ in which } F_{\text{init}} \text{ is greater than (e.g. ten times of) conventional dynamic range mentioned in Proposition 2} \]

\[ \text{i.e. } F_{\text{max}}^{\text{init}} \gg \min_{1 \leq i < \rho} \max(\prod_{s \in i} f_{sr_{r1}}, \prod_{s \in i} f_{sr_{r1}}) \]

\[ \text{Step 1: Categorize moduli of } F'_i \text{ (i.e. } \Gamma = \{f_{sr_{r1}} \ldots f_{sr_{r1}}\}) \text{ to } \rho \text{ disjoint subsets as } a(i, l) \bigcup \bigcup f_{sr_{r1}}, i = 1, \ldots, \rho \text{. The} \]

\[ a(i, l) \text{ is a set of common moduli between } F'_i \text{ and } F_l \text{ and } a(i, l) \text{ is common moduli between } F_i \text{ and all } F_i ' \text{ s. Since } \]

\[ a(i, l) \text{ is obtained by categorizing } \gamma \text{ moduli of } F_i ' \text{, it is possible to write } \bigcup a(i, l) = \gamma, l = 1, \ldots, \rho \].

\[ \text{Step 2: Common compartment modules between } F_i ' \text{ s and } F_i ' \text{ s can be considered as a matrix:} \]

\[ \begin{bmatrix} \end{bmatrix} \]
Based on Step 1 of the procedure each row is related to all $F_i$'s moduli. Thus, each row is chosen such that for $i = 1, \ldots, \rho$
\[ U \cup a(l, i) = \Gamma, l = 1, \ldots, \rho. \]
Each column related to all $F_i$'s moduli. Thus, each column should check to be sure that for $i = 1, \ldots, \rho$
\[ U \cup a(l, i) = \Gamma', i = 1, \ldots, \rho. \]
If this condition is met, go to Step 3; otherwise, return to Step 1 and produce other possible moduli from $\Gamma$.

**Step 3:** Based on (9) and representation $a(l, i)$ in (10) following relationship can be written
\[ F_i - F_i = k(l, i)\text{lcm}(a(l, i)), \]
\[ l = 1, \ldots, \rho and l \neq 1, \ldots, \rho \]

According to lemma 1, the $F_i$ should be zero. By considering $i = 1$ in (11) and $F_i = 0$ we have the following relation:
\[ F_i = k(l, i)\text{lcm}(a(l, 1)), l = 1, \ldots, \rho \]

Now, by substituting (12) in (11), the $F_i$'s for $i > 2$ can be obtained as below:
\[ F_i = k(l, i)\text{lcm}(a(l, 1)) - k(l, i)\text{lcm}(a(l, i)), \]
\[ l = 1, \ldots, \rho and \ l \neq 2, \ldots, \rho \]

that for $i = 1, \ldots, \rho$
\[ k(l, i) \in \{0, \pm 1, \ldots, \pm[F_{\max}/\text{lcm}(a(l, i))]\}, \]
\[ a(l, i) = U \cup f_{\Gamma}, i = 1, \ldots, \rho \]

and also it is assumed that
\[ F_{\max} = F_{\max}^{\text{ini}} \].

**Step 4:** Based on (11), each $F_i$ with each $F_i$ that has common moduli should meet $F_i = k(l, i)\text{lcm}(a(l, i)) + F_i$, similar relations should be met for each $F_i$. When two sets
\[ X = \{F_1, \ldots, F_\rho\} \]
and $Y = \{F_1', \ldots, F_\rho'\}$ are found so that satisfy conditions in (11), we can consider $\max(X)$ as final $F_{\max}$ and finish the process.

Otherwise, choose a bigger $F_{\max}^{\text{ini}}$ e.g. double of previous $F_{\max}^{\text{ini}}$ and go to step 1. It is notable, Proposition 7 presents a relationship to find the largest dynamic range ($F_{\max}$ ) numerically by procedure 1 for any $\rho$ that not presented in the previous studies. However, procedure 1 can be simplified for some cases include $\rho = 2$ and $\rho = 3$. By considering two integers, i.e. $\rho = 2$, we show in Corollary 1 that the close form relationship for the largest dynamic range of two integers in [23] is a special case of proposed proposition 7.

**Corollary 1:** The largest dynamic range (maximum possible range of frequency for unique detection) for proposed proposition 7 when $\rho = 2$ (two frequencies) is
\[ F_{\max} = \min_{l_1, l_2 \in \Gamma} (\text{lcm}(l_1) + \text{lcm}(l_2)). \]

Proof of Corollary 1: For this case, the condition in (9) can be written as
\[ \sum_{l=1}^{2} \sum_{l=1}^{2} F_i - F_j - k(l, i)\text{lcm}(a(l, i) \cup f_{\Gamma}) = 0. \]

Thus, there are the following relationships:
\[ F_1 - F_1 = k_{1,1}\text{lcm}(a(l, 1), \Gamma), \]
\[ F_1 - F_2 = k_{1,2}\text{lcm}(a(l, 2), \Gamma), \]
\[ F_2 - F_1 = k_{2,1}\text{lcm}(a(l, 1), \Gamma), \]
\[ F_2 - F_2 = k_{2,2}\text{lcm}(a(l, 2), \Gamma). \]

Let us show common compartment modules between $F_i$'s and $F_i$'s as a matrix:
\[ F_1 \quad F_2 \]
\[ F_1' \quad F_2' \]
\[ a(l, 1) \quad a(l, 2) \]
\[ a(l, 1) \quad a(l, 2) \]

where $a(l, i)$ is the common disjoint moduli between $F_i$ and $F_i$. Let’s consider community of all moduli as $\Gamma = \bigcup_{l=1}^{\rho} f_{\Gamma}$. Thus, community between subsets of $F_i$ i.e. $a(l, 1)$ and $a(l, 2)$ in a row of mentioned matrix should be $\Gamma$ i.e.
\[ a(l, 1) \cup a(l, 2) = \Gamma \] similar for $F_i$, $F_1$ and $F_2$ there are $a(l, 1) \cup a(l, 2) = a(l, 1) \cup a(l, 2) = a(l, 1) \cup a(l, 2) = \Gamma$. These conditions are satisfied when $a(l, 1) = a(l, 2)$ and $a(l, 1) = a(l, 2)$. In other words, if all modules $\Gamma$ are divided to two disjoint subsets $I_1$ and $I_2$ where $I_1 \cup I_2 = \Gamma$ then the matrix in (15) can be rewritten as:
\[ F_1 \quad F_2 \]
\[ F_1' \quad F_2' \]
\[ a(l, 1) \quad a(l, 2) \]
\[ a(l, 1) \quad a(l, 2) \]

Now, according to (14) and (16) and based on Lemma 1 by considering $F_i = 0$ it can be written:
\[ F_1 = k_{1,1}\text{lcm}(l_1), \]
\[ F_2 = k_{2,1}\text{lcm}(l_2), \]
\[ F_1' - F_2 = k_{1,2}\text{lcm}(l_1), \]
\[ F_2' - F_2 = k_{2,2}\text{lcm}(l_1). \]
To satisfy both relations \( F_2 = k_{1,1}lcm(I_1) - k_{1,2}lcm(I_2) \) and \( F_2 = k_{2,1}lcm(I_2) - k_{2,2}lcm(I_1) \) in (14) it should satisfy \( k_{1,1} = k_{2,1} = 1 \) and \( k_{1,2} = k_{2,2} = -1 \). Thus, \( F_2 = lcm(I_1) + lcm(I_2) \) and the set of integers would be \( X = \{0, F_2\} \) and \( F_{max} \) is the minimum possible of \( F_2 \), i.e.

\[
F_{max} = \min_{i_1 \cup i_2 = \Gamma} \max(X) = \min_{i_1 \cup i_2 = \Gamma} \{lcm(I_1) + lcm(I_2)\}.
\]

This, shows proposition 3 is a special case of proposed proposition 7 when \( \rho = 2 \).

**Corollary 2:** The common moduli between \( F_i \)'s and \( \tilde{F}_i \), i.e. \( a_{i,j} = \bigcup \tilde{f}_s \in (\tilde{f}_j) \), for \( \rho = 3 \) that admit the conditions in proposition 7 can be simplified as follows:

\[
\begin{bmatrix}
F_1 & F_2 & F_3 \\
\tilde{F}_1 & a_{1,1} & a_{1,2} & a_{1,3} \\
\tilde{F}_2 & a_{2,1} & a_{2,2} & a_{2,3} \\
\tilde{F}_3 & a_{3,1} & a_{3,2} & a_{3,3} \\
\end{bmatrix}
\]

\[
F_1' \bigcup I_2 \bigcup I_3 \bigcup I_6 \bigcup I_4 \bigcup I_5 \\
F_2' \bigcup I_3 \bigcup I_4 \bigcup I_2 \bigcup I_5 \bigcup I_1 \bigcup I_6 \\
F_3' \bigcup I_5 \bigcup I_6 \bigcup I_1 \bigcup I_4 \bigcup I_2 \bigcup I_3
\]

where \( I_i, i = 1, \ldots, 6 \) are disjoint subsets and \( \bigcup I_i = \Gamma = \bigcup f_{sr} \).

Note that, corollary 2 substitute the steps 1 and 2 of procedure 1 for the calculation \( a_{i,j} \)'s when \( \rho = 3 \).

**Proof of Corollary 2:** By considering the condition in (9), i.e. \( \sum_{i=1}^{3} \sum_{i=1}^{3} F_i' - F_i - k_{i,j}lcm(\bigcup \tilde{f}_s \in (\tilde{f}_j)) = 0 \), it is possible to write the following relations:

\[
\begin{align*}
F_1' - F_1 &= k_{1,1}lcm(\bigcup \tilde{f}_s \in (\tilde{f}_1)), \\
F_2' - F_2 &= k_{1,2}lcm(\bigcup \tilde{f}_s \in (\tilde{f}_2)), \\
F_3' - F_3 &= k_{1,3}lcm(\bigcup \tilde{f}_s \in (\tilde{f}_3)), \\
F_2' - F_1 &= k_{2,1}lcm(\bigcup \tilde{f}_s \in (\tilde{f}_2)), \\
F_3' - F_2 &= k_{2,2}lcm(\bigcup \tilde{f}_s \in (\tilde{f}_3)), \\
F_3' - F_1 &= k_{3,1}lcm(\bigcup \tilde{f}_s \in (\tilde{f}_3)), \\
F_2' - F_3 &= k_{3,2}lcm(\bigcup \tilde{f}_s \in (\tilde{f}_3)), \\
F_3' - F_3 &= k_{3,3}lcm(\bigcup \tilde{f}_s \in (\tilde{f}_3)).
\end{align*}
\]

Similar to corollary 1, the common compartment moduli between \( F_i \)'s and \( \tilde{F}_i \)'s can be shown as below matrix:

\[
\begin{bmatrix}
F_1 & F_2 & F_3 \\
\tilde{F}_1 & a_{1,1} & a_{1,2} & a_{1,3} \\
\tilde{F}_2 & a_{2,1} & a_{2,2} & a_{2,3} \\
\tilde{F}_3 & a_{3,1} & a_{3,2} & a_{3,3} \\
\end{bmatrix}
\]

(20)

where \( I_i, i = 1, \ldots, 6 \) are disjoint subsets and \( \bigcup I_i = \Gamma = \bigcup f_{sr} \).

To satisfy (21), the common moduli in matrix (20) can be expressed as follows:

\[
\begin{align*}
F_1' \bigcup I_2 \bigcup I_3 \bigcup I_6 \bigcup I_4 \bigcup I_5 \\
F_2' \bigcup I_3 \bigcup I_4 \bigcup I_2 \bigcup I_5 \bigcup I_1 \bigcup I_6 \\
F_3' \bigcup I_5 \bigcup I_6 \bigcup I_1 \bigcup I_4 \bigcup I_2 \bigcup I_3
\]

where \( I_i, i = 1, \ldots, 6 \) are disjoint subsets \( I_i \subset \Gamma \) and \( \bigcup I_i = \Gamma = \bigcup f_{sr} \).

In fact, steps 1 and 2 of procedure 1 can be replaced by corollary 2 for calculation \( a_{i,j} \)'s when \( \rho = 3 \).

**Procedure 2:** The procedure of determination input
frequencies from their under-sampled frequencies for complex waveform. There are some similarities between the procedure of determination of frequencies from under sampled frequencies of a sinusoidal complex waveform (i.e. $\sum_{i=1}^{q} A_i e^{i(2\pi f_i t) + w(t)}$) and the determination of frequencies of real sinusoidal waveform (i.e. $\sum_{i=1}^{q} A_i \cos(2\pi F_i t + \varphi_i) + w(t)$) in [2]. However, should consider the fact that the under-sampled frequencies of the real waveform and the complex waveform are different as follows [2]:

$$f_{u(k,j)} = \begin{cases} (-1)^{\tilde{v}_k}(F_j - m_k f_{sk}) ; & \tilde{v}_k \in \{1,2\} \\ F_j - m_k f_{sk} \text{Complex waveform} \end{cases}$$

(23)

Thus, as can be seen in Fig. 2 (b) the under-sampled frequency curve for complex waveforms is not continues and a few noises or changes in frequency ($F_j$) can cause big change in $f_{u(k,j)}$ and reduce $f_{u(k,j)}$ from maximum to zero or vice versa.

Undersampled frequency $f_u(k,j)$ as a function of $j^{th}$ analog input frequency $F_j \in [0,F_{max}]$ after sampling with the $k^{th}$ sampling frequency $f_{sk}$ from (a) real signal waveform and (b) complex signal waveform.

Step 1: There are $q$ frequencies that are sampled with $p$ ADC’s, thus there are $p \times q$ under-sampled frequencies as $\hat{f}_{u(k,j)}$: $k = 1,\ldots,p, j = 1,\ldots,q$. However, the correspondences between $q$ input frequencies and the $q$ outputs under-sampled frequencies are unknown.

Thus, these $p \times q$ noisy under-sampled frequencies should be divided into $q$ groups with $p$ elements in each group as $\{S_1, S_2, \ldots, S_q\} = \{\{f_{u(1,i)}\} , i = 1,\ldots,p\}, \ldots, \{f_{u(i,j)}\} , i = 1,\ldots,p\}, \ldots, \{f_{u(q,i)}\} , i = 1,\ldots,p\}$. In which the set $S_j = \{f_{u(i,j)}\} , i = 1,\ldots,p\}; j = 1,\ldots,q$ denotes a noisy under-sampled frequencies set that corresponding to the $j^{th}$ input frequency.

Step 2: Determines the distance $DIST_j$ for each set of $S_j = \{f_{u(i,j)}\}$, $i = 1,\ldots,p$, $j = 1,\ldots,q$ as follows:

$$DIST_j = \max \{ DIST_{f_{u(1,j)}}, \ldots, DIST_{f_{u(j,p)}} \}$$

(24)

where the procedure for the computing of $DIST_{f_{u(k,j)}}$ and $F_{est}(f_{u(k,j)})$ for each set of $S_j$ is described as follows:

Step 1: Calculate the frequencies $\hat{F}_{sk}$ in the band $\hat{F}_{sk} \in [0,F_{max}]$ from $\hat{f}_{u(k,j)}$, when sampling frequency is $f_{sk}$ as below:

$$\hat{F}_{sk} = \hat{k}_{sk} f_{sk} + \hat{f}_{u(k,j)} ; 0 \leq \hat{k}_{sk} f_{sk} < F_{max} ; \hat{k}_{sk} = 0,1,\ldots$$

(25)

Step 2: Determine under-sampled frequencies $\hat{f}_{ui}$: $i = 1,\ldots,p, i \neq k$ related to $\hat{F}_{sk}$’s when are sampled with sampling frequencies other than sampling frequency in step 1 i.e. $f_{si}$: $i = 1,\ldots,p, i \neq k$.

$$\hat{F}_i = \hat{k}_i f_{si} + \hat{f}_{u(i,j)}$$

(26)

Step 3: Substitute $\hat{f}_{ui}$: $i = 1,\ldots,p, i \neq k$ with their noisy under-sampled $\hat{f}_{u(i,j)}$: $i = 1,\ldots,p, i \neq k$ in (26). Then calculate the following relationship:

$$\hat{F}_i = \{ \hat{F}_i \} \text{ minimize } \{ |\hat{F}_i - \hat{F}_{si}^*| \}$$

(27)

Note that unlike the under-sampled frequencies of the real waveform in the complex waveform small changes caused by noise can make a big change in the under-sampled frequencies as can be seen in Fig.2 (b). Thus, to substitute $\hat{F}_{ui}$ by $\hat{f}_{u(i,j)}$’s should consider the $\hat{k}_{si}$’s ($\hat{k}_{i}^*, \hat{k}_{i}^*, \ldots, \hat{k}_{p}^*$) and $\hat{F}_{si}$’s.

Step 4: Find the $\hat{k}_{si}$’s that minimize the following relationship and name them as $\hat{k}_{si}^*$:

$$\hat{k}_1^*, \ldots, \hat{k}_i^*, \ldots, \hat{k}_p^* = \{ \hat{k}_1^*, \ldots, \hat{k}_i^*, \ldots, \hat{k}_p^* \}$$

(28)

$$\min_{\hat{k}_1,\ldots,\hat{k}_p} \max \{ |\hat{F}_1 - \hat{F}_p^*|, \ldots, |\hat{F}_1 - \hat{F}_p^*|, \ldots, |\hat{F}_1 - \hat{F}_p^*| \}$$

Definition 1: The maximum distance (DIS) between the
frequencies $F_t^i$ in (27) related to $f_u(k,j)$; \( k = 1, \ldots, p \) is called \( D S_f(u(k,j)) \): \( F_t^i = f_u(k,j) \in S_j \) and defined as follows:

\[
D S_f(u(k,j)) = \max \{ |F_t^i - F_t^j|, \ldots, |F_t^i - F_t^p|, \ldots, |F_t^i - F_t^{i+1}|, \ldots, |F_t^i - F_t^{p-1}| \};
\]

\[
\hat{k}_t^i = \hat{k}_t^{i+1}, \ldots, \hat{k}_t^{p} = \hat{k}_t^{p-1}, i = 1, 2, \ldots, p,
\]

**Step 5:** The estimated input frequency is obtained by mean of the frequencies (\( F_t^i \)'s) that minimize (28) as below:

\[
F_{\text{est}}(f(u(k,j))) = \frac{\sum_{i=1}^{p} F_t^i}{p};
\]

\[
\hat{k}_t^i = \hat{k}_t^{i+1}, \ldots, \hat{k}_t^{p} = \hat{k}_t^{p-1}, i = 1, 2, \ldots, p,
\]

**Step 3:** Obtain the possible input frequencies for state of each set in \( S_j; \ j = 1, \ldots, q \) as \( F_{\text{state}(n)} = \{ F_{\text{est}}(f(u(k,i), s_j)) \} \). \( F_{\text{est}}(f(u(k,i), s_j)) \) was calculated in (30) and \( f_u(k,i) \) is \( i \) th under-sampled frequency of \( j \) th input frequency which minimizes the defined distance in (29) i.e. \( D S_{f_u(k,i)} = \min_{F_f(u(k,i))} (D S_f(u(k,i)), \ldots, D S_f(u(p,i))) \).

**Step 4:** Repeat steps 1 to 3 for all different states of dividing \( p \times q \) under-sampled frequencies to \( q \) groups with \( p \) elements in each group. In other words, steps 1 to 3 should be carried out for \( n = 1, \ldots, (p!)^{q-1} \) different states.

Find the state that has the minimum value of \( D S_{\text{state}(n)} \) as below:

\[
D S_{\text{state}(n')} = \min_{n} (D S_{\text{state}(1)}, \ldots, D S_{\text{state}(n)}), \ldots,
\]

The correct input analog frequencies are obtained based on \( n' \) in (31) as \( F_{\text{state}(n')} \).

**Proposition 8:** The maximum tolerable noise that multiple input frequencies \( F_i \in [0, F_{\text{max}}], i = 1, \ldots, p \) from noisy under-sampled frequencies \( f_u(r,i) = f_u(r,i) + \varepsilon_{(r,i)}, r = 1, \ldots, y \); \( i = 1, \ldots, p \), with sampling frequencies \( \varepsilon_{(r,i)} \) from \( F_{\text{sr}}, r = 1, \ldots, y \) is \( \varepsilon_{\text{max(Tolerable)}} = \frac{\chi_{\text{min}}}{4} \). It is notable \( F_{\text{max}} \) is the largest possible range that obtained in proposition 7 and not a large range as the previous works.

The noise of each under-sampled frequency \( \varepsilon_{(r,i)} \) and the maximum noise of all under-sampled frequencies \( \varepsilon_{\text{max}} \) should be less than the maximum tolerable noise as \( \varepsilon_{(r,i)} \leq \varepsilon_{\text{max}} \leq \varepsilon_{\text{max(Tolerable)}} = \frac{\chi_{\text{min}}}{4} \). Where, in this proposition, the frequency \( f_u(r,i) \) is a noiseless under sampled frequency, \( \varepsilon_{(r,i)} \) is an additive noise, \( \varepsilon_{\text{max}} \) is the maximum value of all \( \varepsilon_{(r,i)} \)'s and,

\[
\chi_{\text{min}} = 4\varepsilon_{\text{max(Tolerable)}} = \min_{f_{\text{st1}}, f_{\text{st2}} < f_{\text{sy}}} \{ |k_{f_{\text{st1}}} - k_{f_{\text{st2}}}|; \ k_{f_{\text{st1}}} \neq 0 \}
\]

\[
1 \leq i_2 < i_2 \leq y
\]

Note that, the \( k_{f_{\text{st}}} \) \( t \) in \( (1,2) \) are some integers in (32) can be selected as \( k_{f_{\text{st}}} \in \{0, \pm 1, \ldots, \pm k_{\text{max}} \} \); \( (k_{\text{max}} - 1)f_{\text{st}} < F_{\text{max}} < k_{\text{max}} f_{\text{st}} \) or \( k_{\text{max}} = \frac{[F_{\text{max}} / f_{\text{st}}]}{k_{f_{\text{st}}} \neq 0} \).

**Proof of Proposition 8:** Consider a frequency \( F \) undersampled with \( r = 1, \ldots, p \) sampling frequency as follows:

\[
F = k_r f_{sr} + f_u(j) = \ldots = \widehat{k}_r f_{sr} + f_u(j) = \ldots = \widehat{k}_p f_{sp} + f_u(p,i)
\]

where \( \widehat{k}_r \) is correct integer that relates noiseless undersampled frequency \( f_u(j) \) to \( F \).

Based on (29) the \( D S_{f_u(k,i)} \) is the distance between \( p \) estimations of \( F_j \) from \( p \) available under-sampled frequencies i.e. \( S_j = \{ f_u(r,j), r = 1, \ldots, p \} \).

Consider the distance \( |\hat{F}_1 - F_{\text{st}}| \) in \( D S_{f_u(k,i)} \) in (29) as \( D_{\text{it}} = |\hat{F}_1 - F_{\text{st}}| \). We prove that \( D_{\text{it}} \), for not incorrect chosen of is greater than \( D_{\text{it}} = |\hat{F}_1 - F_{\text{st}}| \) where \( \hat{F}_1 \) and \( \hat{F}_1 \) are the correct estimated frequency of \( \hat{F}_1 \) and \( \hat{F}_1 \), respectively. In other words \( \hat{F}_1 \); \( i = 1, \ldots, p \) are \( \hat{F}_1 \) that \( f_u(j) \) is noisy under-sampled frequencies and \( k_{f_{\text{st}}} \) are equal to the correct one i.e. \( k_{f_{\text{st}}} \) in (33) or \( \hat{F}_1 = k_{f_{\text{st}}} + f_u(j) \).

We have \( f_u(j) = f_u(j) + \varepsilon_{(r,i)} \), \( f_u(j) = f_u(j) + \varepsilon_{(r,i)} \), \( \hat{F}_1 = k_{f_{\text{st}}} + f_u(j) + \varepsilon_{(r,i)} \), \( \hat{F}_1 = k_{f_{\text{st}}} + f_u(j) + \varepsilon_{(r,i)} \)

and substituting \( f_u(j) - f_u(j) = k_{f_{\text{st}}} - k_{f_{\text{st}}} \) from (33) have the following equation:

\[
D_{\text{it}} = |\hat{F}_1 - F_{\text{st}}| = |(k_{f_{\text{st}}} - k_{f_{\text{st}}})f_{sr} + \varepsilon_{(r,i)} - \varepsilon_{(r,i)}|
\]

Now, there are two states. For the correct estimation we have \( k_{f_{\text{st}} = k_{f_{\text{st}}}} \), \( k_{f_{\text{st}} = 0} \) and \( k_{f_{\text{st}} = 0} \). In (34) can be rewritten:

\[
D_{\text{it}} = |\varepsilon_{(r,i)} - \varepsilon_{(r,i)}| \leq 2\varepsilon_{\text{max}} \quad \text{for the correct estimation}
\]

For incorrect estimation \( k_{f_{\text{st}} = 0} \neq 0 \) and \( k_{f_{\text{st}} = 0} \). Thus, for the incorrect estimation can write:

\[
D_{\text{it}} = |k_{f_{\text{st}}} - k_{f_{\text{st}}} - \varepsilon_{(r,i)} - \varepsilon_{(r,i)}| \geq |k_{f_{\text{st}}} - k_{f_{\text{st}}} - \varepsilon_{(r,i)} - \varepsilon_{(r,i)}| \geq |k_{f_{\text{st}}} - k_{f_{\text{st}}} - 2\varepsilon_{\text{max}}|
\]
Based on (32) we have $|k_rf_{sr} - k_sf_{ss}| > 4\varepsilon_{\text{max}}$. Thus, for the incorrect estimation can write:

$$D_{rs} \geq 4\varepsilon_{\text{max}} - 2\varepsilon_{\text{max}} = 2\varepsilon_{\text{max}}; \quad f \text{ or the incorrect estimation} \quad (37)$$

When one of the $F_i$'s is estimated incorrectly $D_{rk} = |F_i - F_i'| \geq 2\varepsilon_{\text{max}}$ in $\text{DIS}_{f_u(k,i)}$, then $\text{DIS}_{f_u(k,j)} \geq 2\varepsilon_{\text{max}}$ or can write:

$$\begin{cases} \text{DIS}_{f_u(k,j)} \leq 2\varepsilon_{\text{max}} \text{ for the correct estimation} \\ \text{DIS}_{f_u(k,j)} \geq 2\varepsilon_{\text{max}} \text{ for the incorrect estimation} \end{cases} \quad (38)$$

It means by minimizing $\text{DIS}_{f_u(k,i)}$ in (29) as (31) when noises are less than $\varepsilon_{\text{max}}$ (tolerable) (32) the frequencies can be determined uniquely.

**Results and Discussion**

**A. The Maximum Possible Dynamic Range of Under-Sampling Frequency Detection**

To demonstrate the proposed approach, consider the largest dynamic range for $\rho = 2$ input frequencies (integers) with $y = 6$ sensors and sampling frequencies $r_y^{-1}$ (moduli) $\Gamma = \bigcup_{i=1}^{29} f_{sr} = \{3,5,7,11,13,17\}$ Hz. According to Proposition 1, the dynamic range is $F_{\max} = \min_{i \in \Gamma} \text{lcm}(s_{f(r_1)}, \ldots, s_{f(r_6)}) = \text{lcm}(3,5,7) = 105$.

From Proposition 2 the dynamic range is $F_{\max} = \min_{i \in \Gamma} \text{lcm}(s_{f(s_1)}, \ldots, s_{f(s_6)}) = 516$ for $I_1 = \{3,11,17\}$ and $I_2 = \{5,7,13\}$. Based on (17) $F'_1 = k_{1,1}\text{lcm}(I_1)$ and $F'_2 = k_{2,1}\text{lcm}(I_2)$, and $F_2$ can be obtained from two formulas $F_2 = k_{1,1}\text{lcm}(I_1) - k_{1,2}\text{lcm}(I_2)$ and $F_2 = k_{2,1}\text{lcm}(I_2) - k_{2,2}\text{lcm}(I_1)$. Based on Proposition 1 the minimum possible value for $F_2$ that satisfies both formulas are obtained when $k_{1,1} = 1$, $k_{2,1} = 1$, $k_{2,2} = -1$, $k_{2,2} = -1$, $I_1 = \{3,11,17\}$, and $I_2 = \{5,7,13\}$. Thus, $F'_1 = 561$ Hz, $F_2 = 455$ Hz and $F_{\max} = 1016$ Hz. Two sets $X = \{F_1, F_2\} = \{0,1016\}$ and $Y = \{F_1, F_2\} = \{561,455\}$ have the same remainders and $F_{\max} = \text{max}(X) = 1016$ Hz. Similarly, based on Corollary 1 we have $F_{\max} = \min_{i \in \Gamma} \text{lcm}(I_1) + \text{lcm}(I_2)$ where $I_1 = \{3,11,17\}$, $I_2 = \{5,7,13\}$, $F_{\max} = 1016$ Hz. The dynamic range of two integers without conditions on them for the proposed approach and previous works has been shown in Table 2.

As discussed previously Proposition 3 [23] which is just for two integers is a special case of proposed proposition 7 when $\rho = 2$.

<table>
<thead>
<tr>
<th>Approach</th>
<th>Dynamic range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposition 1 [29], [30]</td>
<td>105</td>
</tr>
<tr>
<td>Proposition 2 [19]</td>
<td>561</td>
</tr>
<tr>
<td>Proposition 3 [17], [23], available just for two integers</td>
<td>1016</td>
</tr>
<tr>
<td>Proposition 7 (proposed approach)</td>
<td>1016</td>
</tr>
</tbody>
</table>

Now, consider the largest dynamic range for $\rho = 3$ input frequencies (integers) for moduli $\Gamma = \{3,5,7,11,13,17\}$. Based on corollary 2 each six disjoint partitions of $\Gamma$ (i.e. $\bigcup I_i = \Gamma$ in matrix form in (18) will satisfy (21) or, equivalently, admit the conditions $i \in \bigcup a(I_i) = \bigcup \left(\bigcup f_{sr}^{(i)}(\rho) = \Gamma, i = 1, \ldots, \rho; \bigcup a(I_i) = \bigcup \left(\bigcup f_{sr}^{(i)}(\rho) = \Gamma, i = 1, \ldots, \rho \right)$ in proposition 7. Now, the obtained $a(I_i)$’s by corollary 2 can be used in steps 3 and 4 of Procedure 1 to find $F_{\text{max}}$ in the following. By considering initial largest dynamic range as $F_{\text{max}}^{\text{int}} = 1000$ Hz in procedure 1, we have $I_1 = \{f_{sr}\}$, $I_2 = \{f_{sr}, f_{s2}, f_{s3}\}$, $I_3 = \{f_{sr}, f_{s2}\}$, and $I_4 = \{f_{sr}, f_{s3}\}$. Thus, based on (18) there is following relation:

$$\begin{bmatrix} F_1 & F_2 & F_3 \\ a(1,1) & a(1,2) & a(1,3) \\ a(2,1) & a(2,2) & a(2,3) \\ a(3,1) & a(3,2) & a(3,3) \end{bmatrix}$$

$$\begin{bmatrix} F_1 & F_2 & F_3 \\ F_2' & F_3' & F_3' \\ F_1' & F_2' & F_3' \end{bmatrix} = \begin{bmatrix} f_{s4} & f_{s1} & f_{s3} & f_{s5} & f_{s6} & f_{s5} & f_{s6} \\ f_{s4} & f_{s6} & f_{s5} & f_{s1} & f_{s3} & f_{s2} & f_{s4} & f_{s5} & f_{s6} \\ f_{s2} & f_{s6} & f_{s4} & f_{s1} & f_{s3} & f_{s5} & f_{s6} \end{bmatrix}$$

The $k_{i,j}$’s that satisfy (11) and (19) are as:

$$\begin{bmatrix} F_1 & F_2 & F_3 \\ F_1' & F_2' & F_3' \end{bmatrix} = \begin{bmatrix} k(1,1) & k(1,2) & k(1,3) \\ k(2,1) & k(2,2) & k(2,3) \\ k(3,1) & k(3,2) & k(3,3) \end{bmatrix}$$

$\begin{bmatrix} 1 & 1 & -131 \\ 25 & 3 & -3 \\ 4 & 30 & -2 \end{bmatrix}$

Based on Step 3 of Procedure 1 the $F_1 = 0$ and $F_2 = F_3 = k(1,1)\text{lcm}(a(1,1)) = k(1,1)\text{lcm}(f_{sr}, f_{s1}, f_{s3}) = 1 \times 1016$ Hz.
Determination of the Maximum Dynamic Range of Sinusoidal Frequencies in A Wireless Sensor Network with Low Sampling Rate

\[ \text{lcm}(\{11,3,7\}) = 231 \text{ and } F_1' = 231. \] Other frequencies also is obtained based on Procedure 1. For the obtained \( X = \{F_1, F_2, F_3\} = \{0,10,886\} \) and \( Y = \{F_1', F_2', F_3'\} = \{231,325,340\} \), the largest dynamic range is \( F_{\text{max}} = \max(X) = 886Hz \) which for all \( F_1' \)'s and \( F_1 \)'s should satisfies \((11)\), i.e. \( F_1' - F_i = k_l(l)\text{lcm}(a(l,i)); \ l = 1,\ldots,\rho, l = 1,\ldots,\rho \). As an example, for \( F_2' - F_3' = k_2(l)\text{lcm}(a(2,3)) \) we have \( 325 - 886 = -3 \times \text{lcm}(11,17) \). It is notable that, the large dynamic range by previous studies based on proposition 2 is \( F_{\text{max}} = \min_{s_i \cup \gamma \neq l} \max\{l\text{lcm}(5(3),17),\text{lcm}(5(5),13),\text{lcm}(7,11)\} = 77 \). The dynamic range of three integers without conditions on them for the proposed approach and previous works has been shown in Table. 3.

Table. 3: The dynamic range for three frequencies (integers)

<table>
<thead>
<tr>
<th>Approach</th>
<th>Dynamic range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposition 1 [19], [30]</td>
<td>17</td>
</tr>
<tr>
<td>Proposition 2 [19]</td>
<td>77</td>
</tr>
<tr>
<td>Proposition 7 (proposed approach)</td>
<td>886</td>
</tr>
</tbody>
</table>

For previous works, the large dynamic range for unambiguous reconstruction of input frequencies is \( F_{\text{max}} = 77Hz \) while the large dynamic range obtained by proposed approach is \( F_{\text{max}} = 886Hz \) that is 11.5 times greater than the previous works.

Assume a digital instance frequency measurement (DIFM) equipped to ADCs with sampling rates \( \Gamma = \frac{r}{\gamma} \) \( \cup \ \{3,5,7,11,13,17\} \times 10^7 \times \{30,50,70,110,130,170\} \times MHz \) what is the maximum possible range when 3 input frequencies come simultaneously? Before our work designer could claim designed DIFM guarantees reconstruction 3 simultaneous input frequencies uniquely until \( 77 \times 10^7Hz = 770MHz \) now based on maximum upper bound obtained by proposed Proposition 7 can claim DIFM can reconstruct frequencies uniquely until \( 886 \times 10^7Hz = 8.86GHz \). For the user of DIFM is also important to know for a higher range of frequency can guarantee to reconstruct frequencies.

B. The Under-Sampling Frequency Estimation for Noisy Waveform

This section, simulates the effect of noises on frequency estimations when sampling frequencies are very low.

The simulations are conducted for appropriate and non-appropriate under-sampling frequencies. For the first simulation, the maximum bound of frequencies is considered as a large bound obtained in the previous works for \( \rho = 3 \) input frequencies and low sampling frequencies \( \Gamma = \{3,5,7,11,13,17\}Hz \).

A large bound as shown in Table. 3 for the previous works is 77 Hz. The maximum tolerable noise for this bound based on Theorem 2 of [2] for complex waveform (not real waveform) and for three input frequencies is 0.6.

In this work, we could find the maximum possible range for unique detection of multiple frequencies when sampling with very low sampling frequencies in Proposition 7 that for mentioned sampling frequencies (i.e. \( \Gamma \)) is obtained 886Hz in the previous section. Simulations have been done for 100000 random frequencies per each upper bound of noise for undersampled frequencies. For the previous works a large obtained dynamic range that guarantee of unique detection was 77 Hz. Thus, random input frequencies are chosen in the range \([0,77]\) in Fig. 3. The newly obtained upper bound frequency for unique detection of input frequencies is 886Hz. Consequently, random input frequencies are chosen in the range \([0,886]\) in Fig. 4.

The procedure for detection frequencies was introduced in Procedure 2. The maximum tolerable noise for the proposed approach for three input frequencies and \( \Gamma \) under-sampling frequencies when \( F_{\text{max}} = 886 \) based on proposed Proposition 8 is \( \varepsilon_{\text{max(Tolerable)}} = \frac{\varepsilon_{\text{min}}} {4} = 0.25 \).

Thus, for the proposed approach the maximum unique detectable frequencies and the maximum tolerable frequency noises are 886Hz and 0.25Hz against 77 Hz and 0.6 Hz for the previous works. For non-appropriate undersampling frequencies like \( \Gamma_{\text{non-appropriate}} = \{5,6,8,12,15,18\}Hz \) that are greater than their counterpart sampling frequencies in \( \Gamma \) but the maximum tolerable noise for this set and \( F_{\text{max}} = 886 \) based on proposed Proposition 8 is \( \varepsilon_{\text{max(Tolerable)}} = \frac{\varepsilon_{\text{min}}} {4} = 0 \). Thus, for non-appropriate low sampling frequencies even without the noise we cannot detect frequencies uniquely as shown in Fig. 5.
Fig. 4: Under-sampling frequency detection for noisy under-sampled frequencies of multiple input frequencies with range \([0, 886]\) (more than 11 times greater range than previous studies) and appropriate sampling frequencies \(\gamma\).  

Fig. 5: Under-sampling frequency detection for noisy under-sampled frequencies for multiple input frequencies with range \([0, 886]\) and non-appropriate sampling frequencies \(\gamma_{non-appropriate}\).

**Conclusion**

This study proposed propositions and a procedure to find the largest possible dynamic range for frequencies in a sinusoidal waveform with any number of frequencies for the unambiguous reconstruction of the frequencies of the waveform with very low sampling rates.

Furthermore, the proposed propositions were specified and simplified for waveforms with two and three frequencies and showed that the previous works for the maximum possible range for reconstruction frequencies of waveforms with two frequencies are a special case of our work.

It has been shown that for some cases the proposed approach could achieve 11.5 times greater dynamic range for the unambiguous reconstruction the frequencies of an under-sampled waveform with very low sampling rates.

A procedure for multiple frequencies detection from reminders (under-sampled frequencies) was proposed and maximum tolerable noises of under-sampled frequencies for unique detection were obtained. There are two main disadvantages when using of under-sampling approaches.

When error in under-sampled frequencies is more than tolerable noise the origin frequencies cannot be reconstructed uniquely.

It is also necessary to have a computation unite to reconstruct origin frequencies from under-sampled frequencies. However, using the under-sampling approaches is obligatory in some situations such as the sampling rates of ADCs are very less than the range of frequencies or because of energy consumption or price cannot use more ADCs to cover a high range of frequencies.

In this study, the maximum upper bound for any number of the input frequencies for complex waveform was investigated. In some applications, direct sampling from the real waveform is needed because of hardware limitations.

The relation between actual frequencies and under-sampled frequencies from under-sampled waveform different for complex sampling and real (direct) sampling. Finding the maximum upper bound for any number of input frequencies from directly under-sampled waveform (none complex waveform) is suggested for future work.

**Author Contributions**

Dr. Ali Maroosi. and Dr. Hossein Khaleghi Bizaki suggested the model and innovation of the problem. Simulation has been done by A. Maroosi. The first draft was written by A. Maroosi and reviewed by H. Khaleghi Bizaki.

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**Conflict of Interest**

The authors declare no potential conflict of interest regarding the publication of this work. In addition, the ethical issues including plagiarism, informed consent, misconduct, data fabrication and, or falsification, double publication and, or submission, and redundancy have been completely witnessed by the authors.

**Abbreviations**

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
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<tbody>
<tr>
<td>ADC</td>
<td>Analog-to-Digital Converter</td>
</tr>
<tr>
<td>CRT</td>
<td>Chinese Remainder Theorem</td>
</tr>
<tr>
<td>DOA</td>
<td>Direction of Arrival</td>
</tr>
<tr>
<td>NOMA</td>
<td>non-orthogonal multiple access</td>
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<tr>
<td>UAV</td>
<td>unmanned aerial vehicle</td>
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<tr>
<td>lcm</td>
<td>least common multiple</td>
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<tr>
<td>DFT</td>
<td>discrete Fourier transform</td>
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</table>
References


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